CHAPTER II

Recent Progress in Free Boundary Theory

1. Wakes

The classical theory of Dirichlet and Joukowsky flows concerns fluid motions past obstacles whose location in space is supposed given. A unique and fascinating chapter in fluid mechanics concerns flows whose boundaries are not given, but must be determined from a knowledge of the pressure along them. Today I shall discuss some recent developments in the theory of flows having such "free" boundaries, with especial reference to their physical validity. Because of this emphasis, what I shall say will overlap very little with surveys of the field by von Karman [27] and Weinstein [34], [34′].

I shall begin by a historical résumé of the subject.[1] Free boundary theory was initiated in 1868 by Helmholtz [26] and Kirchhoff [27′]. At that time, it was well-known that the "Dirichlet flow" (Chapter I, Section 5) around a flat plate held broadside in a moving fluid, has the symmetrically disposed streamlines sketched in Figure 6a. This flow gives rise to the two paradoxes of zero resistance (Chapter I, Section 6), and of infinite velocity (hence infinite negative pressure) at the sharp edge.

Kirchhoff had the inspiration to suggest the alternative flow sketched in Figure 6b. He suggested that only part of the flow would be a Dirichlet flow. This part would separate from the plate at its edges S, S', leaving a static *wake* of "dead water" behind the plate, extending to infinity. In this model, the velocity evidently changes abruptly across the boundary

[1] The present material was also presented in large part at the Symposium on Applied Mechanics at Brown University, August 1947, and at the Seventh International Congress on Applied Mechanics in September 1948.

of the wake, giving rise to a *surface of discontinuity*. Helmholtz further assumed that this surface consisted of streamlines, which may thus be called *streamlines of discontinuity*.

In a non-viscous fluid, such a streamline of discontinuity would give rise to no (viscous) shear forces; for dynamical equilibrium one thus requires that the pressure be continuous

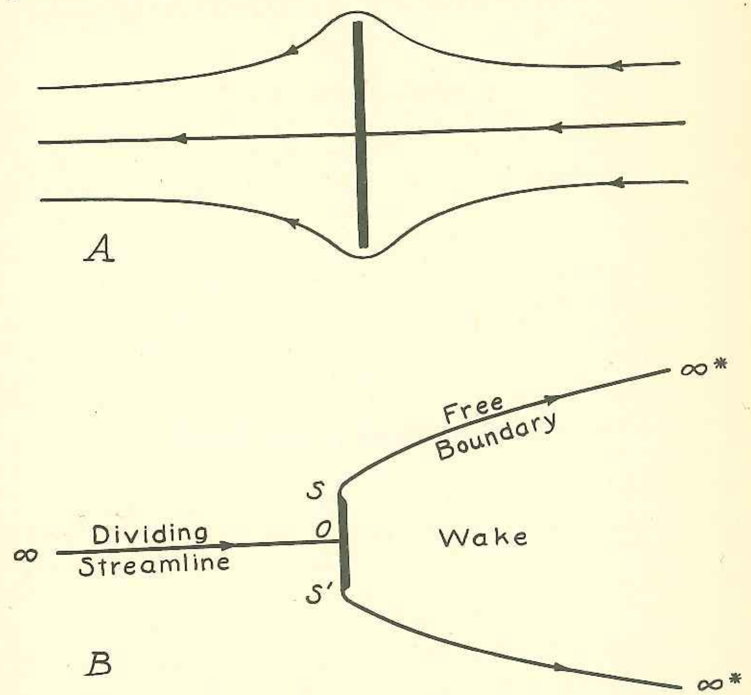

Fig. 6. Flows past vertical plate.

across the streamline. But in the static wake (neglecting gravity, by Chapter III, Theorem 4) the pressure is constant. Hence it is necessary and sufficient for the streamline of discontinuity to be at *constant pressure*. Such a streamline is called a *free streamline*. By Bernoulli's Theorem (Chapter I, Section 6), since $vdv = -dp/\rho = 0$, *velocity is constant* on any such free streamline.

Hence the velocity is finite at S, S', and the paradox of infinite velocity is avoided. We shall see shortly (Section 2) that the d'Alembert Paradox of zero resistance is also avoided. Moreover separation *does* occur physically at the plate edges S, S'. Finally, the model of a "static wake" accords with our qualitative experience that one can find shelter from the wind on the leeward side of a barrier. Thus the Helmholtz model of Figure 6b represents, in the case of a flat plate at least, enormous progress over the Euler-Dirichlet model of Figure 6a.[2]

2. *Mathematical Model*

Following Kirchhoff (cf. [27′] or [10], p. 299) one can obtain a complete mathematical description of the flow of Figure 6b, as follows.

The velocity potential $U(x,y)$ of a locally irrotational plane[3] flow is harmonic by Chapter I, Section 5; hence $U(x,y)$ is the real part of a complex function $W = U + iV$ of the complex *position* coordinate $x + iy$. The function W so defined is called the *complex potential*, and V the *stream function* (it is easy to show that $V =$ const. on streamlines). Moreover dW/dz is the complex conjugate $\zeta = u_1 - iu_2$ of the complex velocity vector $u_1 + iu_2$.

The conditions that the flow be *irrotational* and *volume-preserving* are thus precisely that

$$\frac{dW}{dz} = \zeta, \tag{1}$$

in the sense of complex differentiation. Hence the functional relationships between the complex variables W, ζ, and z are *analytic* in the usual sense, and are therefore expressible by *conformal* transformations. We consider the transformation of the ζ-domain (or "hodograph") onto the W-domain; this may be determined in the present case as follows.

[2] In line with Chapter I, it is however interesting to recall the Hadamard Paradox, whereby an (analytic) surface of discontinuity cannot be generated (J. Hadamard, *Leçons sur la propagation des ondes et les équations de l'hydrodynamique*, Paris 1903, Note II). See also F. Klein, *Zeits. Math. Phys.*, *58* (1910), p. 259.

[3] That is, flow in which the velocity vector is always parallel to a fixed plane, so that $\partial U/\partial z = 0$ and $U = U(x,y)$.

Let the origin be placed at the stagnation point O in the z-plane, and the real axis along the plate SOS'; thus we imagine Fig. 6b rotated through 90° clockwise. Clearly ζ is real along the plate, while

$$|\zeta| = |\zeta(\infty)| = \text{const. on the free streamline.} \tag{2}$$

Hence, under the simplest topological assumptions, the hodograph is a *semicircle*, as in Figure 7. By proper choice of units, $|\zeta(\infty)| = 1$, and so $\zeta(\infty) = e^{i\alpha}$. Thus the hodograph is a unit semicircle, centered in the origin. In the present case of normal incidence, $\alpha = \pi/2$ and so $\zeta(\infty) = i$, as in Figure 7.

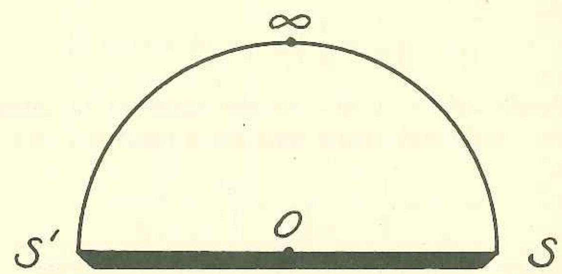

FIG. 7. Hodograph of Kirchhoff flow.

The W-domain is clearly a plane, cut along the positive real axis, setting $W(0) = 0$ by choice of the constant of integration in (2).

By elementary conformal transformations, we can map the hodograph onto the W-domain. Clearly $-\sqrt{W}$ covers the lower half-plane; $\sigma = \frac{1}{2}(\zeta + 1/\zeta)$ maps the hodograph onto the lower half-plane. Hence $-\sqrt{W} = (A\sigma + B)/(C\sigma + D)$, with A, B, C, D real and $AD \neq BC$, also is a one-one conformal ("schlicht") transformation mapping the hodograph onto the W-domain. And it is a classical theorem of conformal mapping that there are no others.

Further, in the present case, $W = \sqrt{W} = 0$ implies $\zeta = 0$ and $\sigma = \infty$; hence $A = 0$, and we can write $-\sqrt{W} = 1/C\sigma + D)$. Finally, $W = \infty$ implies $\zeta = i$ and so $\sigma = 0$; hence $D = 0$ in the case of normal incidence. Substituting, we get

$$cW = \frac{c}{C^2\sigma^2} = \left[\frac{\zeta}{\zeta^2+1}\right]^2, \quad \text{if} \quad c = \frac{C^2}{4}. \tag{3}$$

All choices of the constant c in (3) give similar flows; a typical solution is given by setting $c = 1$. In this case, the computation of $z(\zeta)$ reduces by (1) to the integration of a rational function. Performing this integration, we get

$$z = \frac{\zeta}{(\zeta^2+1)^2} + \frac{1}{2}\left\{\frac{\zeta}{\zeta^2+1} - \frac{i}{2}\ln\frac{\zeta - i}{\zeta + i}\right\} + \text{const.}, \tag{4}$$

for all values of $\zeta = \xi + i\eta$.

Along the plate, ζ is real and (4) reduces to

$$z = \frac{\zeta}{(\zeta^2+1)^2} + \frac{1}{2}\left\{\frac{\zeta}{\zeta^2+1} + \arctan\zeta\right\}, \tag{4'}$$

where clearly $z(0) = 0$, and so the constant of integration vanishes. At the right-hand separation point S, $\zeta = 1$ and so

$$z(S) = \frac{1}{4} + \frac{1}{2}\left\{\frac{1}{2} + \frac{\pi}{4}\right\} = \frac{4+\pi}{8}. \tag{4''}$$

The pressure is most easily computed by setting $\zeta = \xi = \tan\theta$ along the plate, so that $\zeta/(\zeta^2+1) = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$. Hence the thrust on half the plate is, by Bernoulli's Theorem and (1),

$$\int_{\xi=0}^{1}(1-\xi^2)dx = \int_{\xi=0}^{1}\frac{(1-\tan^2\theta)}{\tan\theta}\, d\left[\frac{1}{4}\sin^2 2\theta\right] = \int_{\theta=0}^{\pi/4} 2\cos^2 2\theta d\theta = \frac{\pi}{4} \tag{5}$$

The ratio of half the thrust to the plate half-length is clearly the *drag coefficient*, which is thus

$$C_D = \frac{2\pi}{(\pi+4)} = 0.88 \quad \text{approximately}. \tag{6}$$

Similar but more elaborate complications give, for the case of *oblique* incidence $\alpha \neq \pi/2$,

$$C_D = \frac{2\pi\sin^2\alpha}{4+\pi\sin\alpha}, \qquad C_L = \frac{\pi\sin 2\alpha}{4+\pi\sin\alpha}. \tag{6'}$$

Details may be found elsewhere ([9], p. 103; [10], pp. 304–305; [13], Vol. I, p. 287). Figure 2c shows the flow for the case $\alpha = 15°$.

3. *Borda Mouthpiece*

A similar procedure gives a plane *jet* from an infinite straight orifice (Borda tube), as sketched in Figure 8a; the dotted area is supposed to consist of static air. The problem is mathematically simplified if we consider only the half of the flow above the axis of symmetry; in this form, the problem was solved by Helmholtz in 1869.

If we take the axis of symmetry as the real axis, and choose units so that the jet velocity is one, we again have a semicircular hodograph as in Figure 7. Hence $\sigma = \frac{1}{2}(\zeta + 1/\zeta)$ again fills the lower half-plane. The difference is that the present flow is bounded by *two* streamlines which (by choice of units) we can take to bound the *strip* $0 \leqq V \leqq \pi$ in the W-plane. As before, we thus get $e^W = (A\sigma + B)/(C\sigma + D)$.

We again calculate A, B, C, D by the method of undetermined coefficients. Thus when $W = -\infty$, and $e^W = 0, \zeta = 0$ and so $\sigma = \infty$; hence $A = 0$ and we can set $B = 1$ as before. Whereas at $W = +\infty$, $e^W = \infty$ and $\zeta = \sigma = 1$; the asymptotic jet velocity; hence $D = -C$. Substituting, we get

$$Ce^W = \frac{1}{\sigma - 1} = \frac{2\zeta}{\zeta^2 - 2\zeta + 1} = \frac{2\zeta}{(\zeta-1)^2} \qquad [C \text{ real}] \tag{7}$$

We get in this way a family of flows, all similar to the flow

$$W = \ln\zeta - 2\ln(\zeta - 1) + \text{const.} \tag{7'}$$

The determination of z then reduces to

$$z = \int\frac{dW}{\zeta} = \int\frac{d\zeta}{\zeta^2} - 2\int\frac{d\zeta}{\zeta(\zeta-1)} = -\frac{1}{\zeta} + 2\ln\zeta - 2\ln(\zeta-1) + \text{const.} \tag{8}$$

as can easily be shown using partial fractions. We can make z real when $\zeta \downarrow 0$ through positive real values, by making the integration constant $2\ln(-1)$, so that (8) becomes

$$z = -\frac{1}{\zeta} + 2 \ln \frac{\zeta}{1-\zeta} = -\frac{1}{\zeta} + 2 \ln \zeta - 2 \ln (1 - \zeta) \tag{8'}$$

Again, as $\zeta \uparrow 0$ through negative real values at ∞^*, $y = Im(z) \to \pi$. Hence the walls of the Borda mouthpiece are 2π units apart. Finally, as $\zeta = e^{i\theta} \to 1$, along the free streamline $Im(-1/\zeta + 2 \ln \zeta) \to 0$, while the imaginary part of $\ln (1 - \zeta)$ approaches $\pi/2$. Hence *the jet width π is half the*

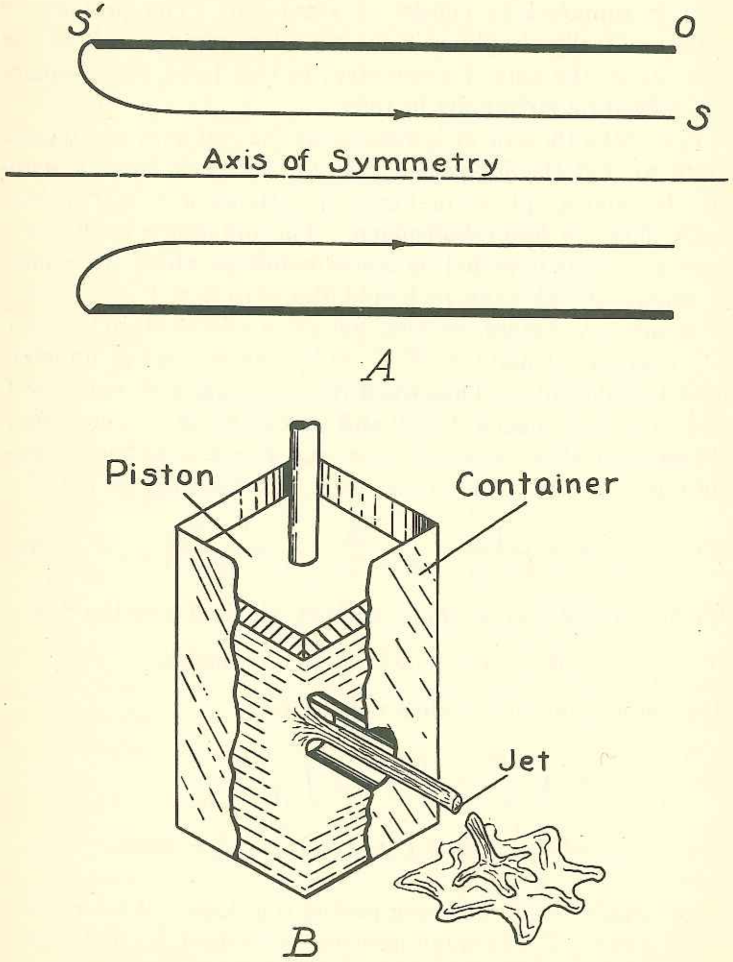

FIG. 8. Borda flows through slot and circular orifice.

orifice width. In other words, the "coefficient of contraction" of the orifice is one-half.

The coefficient of contraction (but not the velocity field) can also be obtained by simple energy and momentum considerations, as follows (cf. [12], Vol. I, p. 242.) Let a container with vertical walls be filled with a liquid of density ρ, into which is inserted a horizontal "Borda tube" as in Figure 8b, of arbitrary cross-section shape and area A; let the pressure head at the level of the tube be p. We assume that the flow separates[4] from the tube at its inner end, and that the jet from the tube tends asymptotically to a constant speed v; this must be the constant "free streamline" velocity on the jet boundary. Let A^* be the asymptotic jet cross-section; then A^*/A is by definition the coefficient of contraction. We compute it as follows.

Per unit time, the volume efflux is vA^*; the momentum efflux is v^2A^*; the kinetic energy efflux is $\frac{1}{2}\rho v^3 A^*$. The momentum supplied is however pA (pressure excess); the (potential) energy $p(vA^*)$. Hence $pA = \rho v^2 A^*$ and $pvA^* = \frac{1}{2}\rho v^3 A^*$. Dividing v times the first equation by the second, we get the result

$$\frac{A^*}{A} = \frac{1}{2}. \tag{9}$$

This conclusion is not restricted to plane flows, but is clearly equally valid for a circular tube.

4. Further Generalizations

Many other models of wakes and jets with free streamlines have been constructed since the time of Helmholtz and Kirchhoff. They are fully described in the classical literature,[5] which has become known as the theory of wakes and jets.

A large proportion of these flows involve cases where the diagram in the ζ-plane, or hodograph, is a circular sector. It

[4] This is the case of "free flow"; see A. H. Gibson, *Hydraulics*, fourth ed., p. 122; it is not always fulfilled.

[5] See [9], pp. 94–109; [10], Chs. XI–XIII; [25], [31], and [32]. A complete survey of the subject is being prepared by the author and E. Zarantonello under the title "Jets, wakes, and cavities."

has recently been shown[6] that $z(\zeta)$ can be expressed explicitly in terms of elementary functions, whenever this circular sector subtends a rational angle (i.e. one commensurable with 360°).

Indeed, in this case $\sigma = \zeta^{m/n}$ can be made to cover the upper half of the unit circle $|\zeta| \leqq 1$, for some pair of integers m, n (in the case of free streamlines, we easily reduce to a sector of the unit circle). Hence $\tau = \frac{1}{2}(\sigma + 1/\sigma)$, which is a rational function of $\sqrt[n]{\zeta}$, covers the lower half plane.

On the other hand, a stationary flow must be bounded by streamlines or dividing streamlines. Hence the diagram in the W-plane (*if* simply connected) consists of a plane, half-plane, or strip, possibly with a number of semi-infinite cuts parallel to the real axis. Therefore there is a Schwarz-Christoffel transformation of the σ-domain onto the W-domain of the form $dW = F(\tau)d\tau$, where $F(\tau)$ is a *rational* function; this follows immediately from the theory of Schwarz-Christoffel transformations.

Expressing $F(\tau) = F(\frac{1}{2}(\zeta^{m/n} + \zeta^{-m/n})) = R(\sqrt[n]{\zeta})$ as a rational function of $\sqrt[n]{\zeta}$, we obtain by substitution in (1)

$$(10) \quad z = \int \frac{dW}{\zeta} = \frac{m}{2} \int R(\sqrt[n]{\zeta})((\sqrt[n]{\zeta})^m - (\sqrt[n]{\zeta})^{-m}) \frac{d\sqrt[n]{\zeta}}{(\sqrt[n]{\zeta})^{n+1}}$$

The integrand is evidently rational in $\sqrt[n]{\zeta}$, whence z, as well as ζ and W, can be expressed as a sum of rational and logarithmic complex functions of $\sqrt[n]{\zeta}$. I omit the details, which show that the breaking up of the integrand in (10) into partial fractions, can indeed be computed explicitly in all cases.

Although the procedure just outlined is attractive in principle, it involves serious practical difficulties. Unless m/n is $\frac{1}{2}$ or 1, the final formulas are very long. Hence, unless one is satisfied with the relation between a few physical dimensions, it is better to evaluate z by numerical integration than by partial fractions.[7]

Using complex numerical integration, it has proved possible

[6] Birkhoff, Plesset, and Simmons [20]. The case where the diagram in the W-plane is an infinite strip had already been covered by Mises ([9], Appendix of second German edition).

[7] A detailed account of some difficulties of numerical calculation has been given by D. M. Young, "The use of conformal mapping to determine flows with free streamlines," *Proc. Symposium on Construction and Applications of Conformal Maps*, Inst. Num. Analysis, Nat. Bureau Standards, Univ. Calif. at Los Angeles, June, 1949.

to code the calculation of flows with free streamlines whose hodograph is a circular sector for the Mark II Calculator at Dahlgren, so that the velocity field over most of the flow can be computed within hours.[8]

Similar arguments enable one to express flows past general polygonal barriers in terms of hyperelliptic integrals, using $\omega = \ln \zeta$ as independent variable. (Clearly the hodograph is a rectangular polygon in the ω-plane.)

An even more important result is due to Levi-Civita [29], who used the Schwarz Principle of Reflection to express the most general flow having an infinite wake, of the topological type discussed in Section 2, in terms of analytic functions. By this method, one can easily construct flows, with "wakes" bounded by free streamlines, past *curved* obstacles. However, the shape of the obstacle cannot be prescribed in advance, except qualitatively; thus the effective determination of the plane flow or flows with "wake" past a *given* curved obstacle remains an unsolved problem.[8a]

5. *Indeterminacy of Separation Point*

In the cases treated in Sections 2 and 3, the topology of the cavity and the exact location of the points of flow separation were assumed known. These assumptions are not possible in general, and lead to serious problems of *mathematical indeterminacy*.

Thus H. Villat proved, thirty-five years ago, that even in the simple case of flows with infinite wake past a reentrant wedge, separating at the edges, more than one topological configuration was mathematically possible;[9] specifically, a dead

[8] The coding and method of calculation were planned by Prof. D. R. Hartree, Mr. J. C. Aller, and the author; useful suggestions were also received from Prof. John von Neumann.

[8a] For exceptions, see S. Brodetsky, *Proc. Zurich Congress Appl. Mech.* (1926), 527–531.

[9] *Loc. cit.* on p. 21; see also *Ann. Fac. Sci. Toulouse*, *5* (1913), 375–404; *Ann. Ec. Norm. Sup.*, *31* (1914), 455–491; *Bull. Sci. Math.*, *42* (1918), 72-92; *Apercus théoriques sur la résistance des fluides*, Paris, 1920; R. Thiry, *Ann. Éc. Norm.* Sup. 38 (1921), 229–339.

area at arbitrary pressure can form in the reentrant vertex. This result has recently been extended by Dr. E. Zarantonello, who has shown that, in the case of a plate with angle of attack $\alpha \neq 0°$, exactly two topological types of flow with infinite wake and everywhere finite velocity are possible.

A more fundamental indeterminacy arises in the case of curved obstacles, even for flows having the topological type discussed in Section 2. It was first noted in 1911 by Marcel Brillouin ([22], p. 174) that "la forme de M. Levi-Civita est encore beaucoup trop générale." Brillouin also

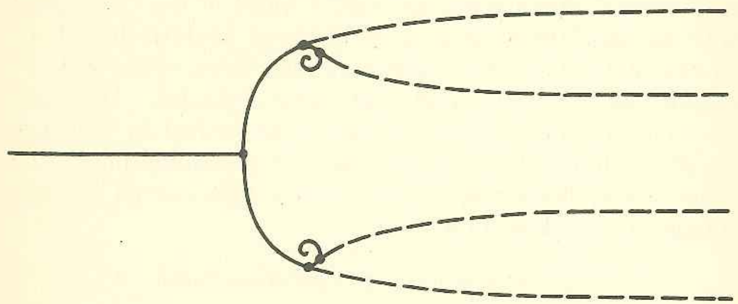

FIG. 9. Two cavity flows past accolade.

suggested most of the criteria mentioned below, for avoiding this indeterminacy. As the existence and uniqueness theory has been summarized recently elsewhere ([34] and [34′]), I shall only state the most essential facts. These are clearly illustrated by the following fundamental theorem of Leray [28], culminating earlier work of Weinstein and Friedrichs (see [34], [34′]).[10]

Let B be any circle, or more generally any symmetric convex barrier whose curvature is non-decreasing as one moves backwards from the stagnation point, as in Figure 9. Such barriers are called "accolades."[11] Then there is a fixed point S_0 on B,

[10] Leray's results were extended to jets and to wakes in channels by J. Kravtchenko, *Jour. de Math.*, *29* (1941), 30–303; also *Ann. Inst. Henri Poincaré*, *62* (1945), 233–268. See also H. Villat, *Jour. math. pures appl.*, *10* (1914), 231–290; S. Brodetsky, *Proc. Edin. Math. Soc.*, *41* (1922–1923), 58–62.

[11] By the four-vertex theorem, the circle is the only "accolade" which bounds a convex plane region. However, other generalizations are possible.

such that for every point S behind S_0 on B, exactly one symmetric flow with infinite cavity (wake) is possible which separates from B at S.

By this theorem, there is a one-parameter family of cavity flows past B with infinite wakes. The free streamlines corresponding to two such flows are sketched in Figure 9. The shape of each wake is asymptotically parabolic, in the sense that

$$(11) \quad y \sim \sqrt{Cx} \qquad \text{for some } C; \qquad \text{i.e., } \lim_{x\to\infty} \frac{y}{\sqrt{Cx}} = 1.$$

The "drag" D is determined by C through the formula

$$(11') \qquad D = \frac{\pi\rho C}{4},$$

of Levi-Civita [29]. (Similar formulas for the lift and moment of asymmetrical cavity flows were obtained by M. Brillouin [22].)

One of these flows is preferable esthetically, on the following mathematically equivalent grounds: (i) separation occurs at S_0, (ii) the minimum pressure occurs in the cavity (i.e. at the point of separation), (iii) the cavity is convex, (iv) the curvature of the cavity at the separation points is finite.

Although mathematicians have tended to stress condition (iv), it seems to me that condition (ii) is far more significant physically. Thus physically, it is presumably fulfilled in the case of a vapor-filled cavity,[11a] or if $p \ll \frac{1}{2}\rho v^2$. In fact, I know of only one physically plausible case[12] in which it is not satisfied. Also, though (ii) always implies (iii), as noted by Brillouin, condition (iv) is not usually satisfied when separation occurs at the *end* of a polygonal or other barrier, whereas (ii) is.

Condition (ii) is, of course, not fulfilled for physical wakes

[11a] In ordinary liquids, the pressure cannot long remain sensibly below vapor pressure. The contrary assertion of Rayleigh ([13], Vol. 6, p. 401) applies only to liquids whose "gas nuclei" have been removed by boiling or other special treatment; cf. [22], p. 151. For direct confirmation, see [15], Fig. 34; also Section 9.

[12] The case described by M. J. Lighthill, *Aer. Res. Comm. Reps. and Mem.*, *2105*, of a jet sucked around a bend.

behind curved obstacles ([6], Vol. 2, pp. 422, 497), but neither is any other part of classical "wake theory."

6. *Physical Unreality*

Indeed, physical unreality is well-known to be one of the three capital defects of the classical theory of free streamlines. (The other two are, that it is mathematically indeterminate in general, and can be applied only to plane flows—and practically only to a few plane flows past polygonal fixed walls.)

The physical unreality of the Kirchhoff wake model has been realized for over fifty years.[13] It may be shown (using plausible, if not strictly rigorous arguments!) that a streamline of discontinuity in a non-viscous fluid is in *unstable* equilibrium. It may also be observed experimentally that it soon becomes wavy, and then breaks up into a turbulent "mixing zone," which has been discussed by various writers.[14] This turbulence is drawn into the wake and recirculated there, so that real wakes are by no means motionless bands of "dead water" extending to infinity. At lower Reynolds numbers ($50 < R < 10^5$ for cylinders), they often consist of alternate vortices or "vortex streets."[15]

Some idea of the magnitude of the discrepancy between the Kirchhoff model and real wakes may be had from the fact that the real C_D of a flat plate is about 1.95 ([14], p. 227), or twice the $C_D = 0.88$ predicted by theory. Whereas the theoretical wake pressure is equal to the free stream pressure, *there is a large actual wake underpressure.* This wake underpressure causes the actual "streamlines of discontinuity" to

[13] See Kelvin (*Nature, 50*, 1894, pp. 524, 549, 573, 597); also Rayleigh [13], Vol. 6, p. 39. Prophetically, the instability of a surface of discontinuity was discussed (in another connection) by Kelvin in 1871 (*Phil. Mag.*, Vol. 42, p. 368); see also Rayleigh [13], Vol. 1, pp. 365, 477; F. W. Lanchester *Aerodynamics* (1908), pp. 128–136.

[14] See [6], Sections 252–254; also L. M. Swain, *Proc. Roy. Soc., A125* (1929), p. 647; S. Tomotika, *ibid., A165* (1938), pp. 53–72; L. Schlichting, *Ing. Archiv. I* (1930), p. 533; W. Tollmien, *ibid. 4*, (1933), pp. 1–15.

[15] See Th. von Karman and H. Rubach, *Phys. Zeits., 13* (1911), p. 49; M. Teissié-Solier and P. Dupin, *Les tourbillons alternés*, Paris, 1928; L. Rosenhead and M. Schwabe, *Proc. Roy. Soc., A129* (1930), pp. 115–135.

lie inside those predicted by Kirchhoff.[16] It also causes the observed lift coefficient C_L of a tilted plate to greatly exceed that calculated on the basis of "wake theory." This underestimate led to undue pessimism in the early days, concerning the possibility of flight by heavier-than-air craft.

Since the magnitude of the wake underpressure appears to persist no matter how small the viscosity is, it seems to me that the flaw in theory is not due so much to physical as to mathematical oversimplification (lack of mathematical rigor in treating the case of negligible viscosity). I shall therefore refer to the Paradox of Wake Underpressure.

A similar discrepancy between theory and observation exists for *jets* flowing out of an orifice into a fluid of equal density. So far as I know, good agreement between classical free streamline theory and experiment has been found only in the case of liquid jets issuing into air, or some other low density medium.[17]

7. *Dimensionless Parameters ρ'/ρ and Q*

A familiarity with the preceding facts leads one to focus attention on two physical parameters which are completely ignored in the classical theory of jets and wakes. Thus neither is even mentioned in the expositions of [9], [10], [12], [34], or [34′]; much of the recent progress which I shall report depends on recognizing their role explicitly.

The first of these parameters is the dimensionless ratio ρ'/ρ of the densities of the fluids on the two sides of the streamline of discontinuity. It is 1.0 in classical wake theory, and about .0013 for a water jet in air.

[16] O. Flachsbart, *Zeits. ang. Math. Mech., 15* (1935), pp. 32–37; [6], p. 554; Camichel, *Comptes Rendus, 174* (1922), 666–669; L. Escande and M. Teissié-Solier, *ibid., 191* (1930), p. 519. For aeronautical implications, see Lanchester, *op. cit.* supra, pp. 264–265; Kelvin, *Collected Papers*, Vol. 4, 205–210.

[17] For cases of such agreement, see [13], Vol. 1, p. 304; R. von Mises, *Zeits. Ver. Deutsche Ing., 61* (1917), 447–451, 469–473, and 493–497; [16]. For jets flowing into media of the same density, see L. Marty, *Ann. Fac. Sci. Toulouse, 22* (1930), 41–146; E. Foerthmann, *Ing. Archiv, 5* (1934), 42–54; [6], Section 255; D. Citrini, *Energia Elett., 23* (1946), 133–144 and 302–315, and *24* (1947), 177–192; H. Rouse, *Proc. Am. Soc. Civ. Eng., 74* (1948), 1571–1596.

Thus the following plausible conjecture was made in 1931 by Betz and Petersohn [16]: *if ρ'/ρ is small, then classical free streamline theory can be applied.* (It is to be noted that the smallness of the kinematic viscosity is not mentioned). Now ρ'/ρ is small for jets of liquid issuing into a gas, and again in the so-called *cavity motion* which occurs behind high-speed missiles in water. In the latter case (cf. [19], [23], [35]) a "cavity" filled with low-density air or water vapor (ρ'/ρ = .00003) fills the region behind the missile. Hence it is plausible that classical "wake theory" may apply to cavity motion—it certainly does not apply to wakes!

We shall see later (Section 10) that even the conjecture of Betz and Petersohn oversimplifies the facts. Nevertheless, it represents an important step forward, and I shall speak from now on of *jets and cavities*, instead of using the traditional "jet and wake" terminology. Any "jets" to which I refer should also be thought of as liquid jets moving through air or some other low-density medium.

The other is the dimensionless ratio expressing the relative wake underpressure: the so-called *cavitation number*

$$Q = \frac{(p - p_c)}{\frac{1}{2}\rho v^2}. \tag{12}$$

Here p denotes the "free stream pressure" of the undisturbed flow; v is its relative velocity; ρ is the fluid density, and p_c is the "wake pressure" on the streamline of discontinuity. In the classical theory, $Q = 0$ is assumed, corresponding to infinitely long cavities.[18]

Now physical cavities usually have finite length and positive cavitation number $Q > 0$; moreover the minimum pressure usually occurs inside the cavity. It was observed by M. Brillouin [22] that these conditions were incompatible with irrotational flow in an ideal fluid, since they would lead to a convex cavity. Such a cavity would have a stagnation point

[18] As pointed out to me orally by G. Kreisel, an infinitely long cavity in an infinite stream with $Q \neq 0$ is incompatible with regular flow at infinity, by the Phragmèn-Lindelöf Principle. I note also that, by the underwater explosion paradox of Chapter I, Section 17, a cavity cannot form in plane flow.

at the rear end of the free boundary, at minimum pressure p_s. By Bernoulli's Theorem, $v^2 = p_s - p \leqq 0$ therefore, which implies no motion at all; this is the Paradox of Brillouin.

One can escape from this paradox, and get at least a partial mathematical approximation to physical reality, by supposing that artificial alterations of the rear of a cavity, far from an obstacle, do not greatly affect cavity flow about the obstacle. I shall now discuss some such mathematical approximations.

8. *Flows with $Q \neq 0$*

In 1921, Riabouchinsky [32] constructed hydrodynamic flows with free streamlines about two symmetrically placed

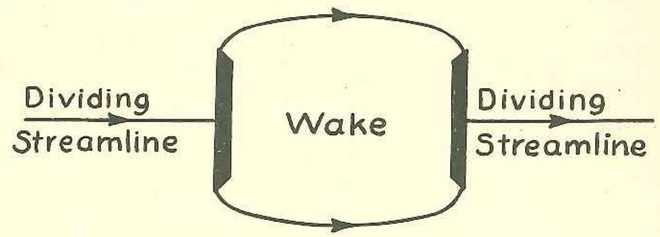

FIG. 10. Riabouchinsky flow past plate.

plates (see Fig. 10), with $Q > 0$. A sketch of Riabouchinsky's mathematical[19] discussion follows.

The "hodograph" (i.e. diagram in the ζ-plane) of one quarter of the flow is clearly a quarter-circle, and the W-domain is a quadrant, by vertical and horizontal symmetry. Hence the conformal map of the hodograph on the W-domain satisfies, much as in Section 2,

$$W^2 = \frac{A(\zeta^2 + \zeta^{-2}) + B}{C(\zeta^2 + \zeta^{-2}) + D} = c\,\frac{\zeta^4 + 2\lambda\zeta^2 + 1}{\zeta^4 + 2\mu\zeta^2 + 1}. \tag{13}$$

By choice of units, we can reduce to the case $c = 1$, and $\zeta = 1$ at $W = \infty$, which implies $\mu = -1$. This gives

[19] Physically, Riabouchinsky's original discussion referred to wakes, where we should refer to cavities; see *Proc. III Int. Congr. Appl. Mech. Stockholm* (1930), Vol. 1, 137–158. The relevance to cavitation with $Q > 0$ was noted by F. Weinig, *Hydromechanische Probleme des Schiffsantriebs*, Hamburg, 1932, 294–300.

$$W = \frac{\sqrt[2]{\zeta^4 + 2\lambda\zeta^2 + 1}}{(\zeta^2 - 1)}, \tag{14}$$

from which $z = \int dW/\zeta$ can be obtained as an elliptic integral. If v is the free streamline velocity, then $\zeta = v$ when $W = 0$, whence $\zeta^4 + 2\lambda\zeta^2 + 1 = (\zeta^2 - v^2)(\zeta^2 - v^{-2})$, and $\lambda = -\frac{1}{2}(v^2 + v^{-2})$. By Bernoulli's equation, $Q = v^2 - 1$, whence $\lambda = -\frac{1}{2}[(1 + Q) + (1 + Q)^{-1}]$. With these formulas, it is easy[20] to obtain C_D as a function of Q.

Another important model for cavity flow with $Q > 0$ has been constructed recently by Efros, Gilbarg and Rock.[21]

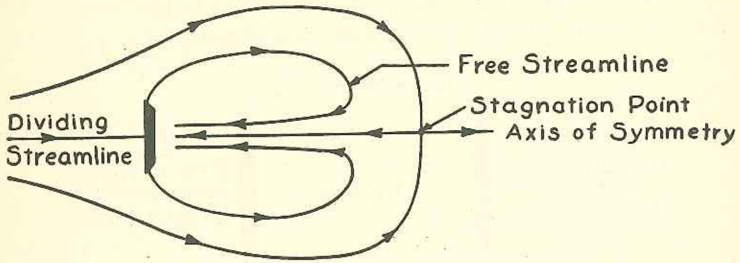

FIG. 11. Cavity flow with reentrant jet.

This involves a *reentrant jet* (see Fig. 11), instead of a symmetric cavity. Such jets, somewhat analogous to those discussed later (Sections 12–13), are now known to occur experimentally when a symmetrical object is dropped into water. It is therefore likely that this model closely resembles actual cavity motion, though the two-sheeted Riemann surface required for the z-plane cannot of course be realized physically.

The Riabouchinsky and Gilbarg models agree with each other in two respects. In both models, C_D increases with Q in the approximate ratio

$$C_D(Q) \cong (1 + Q)C_D(0) = 0.88(1 + Q) \tag{15}$$

[20] In 1946, T. E. Caywood extended Riabouchinsky's analysis to the case of a wedge in a finite channel, but the numerical computations involve untabulated definite integrals.

[21] D. Gilbarg and D. H. Rock, *Naval Ordnance Laboratory Memo., 8718* (1945); D. A. Efros, *Doklady Akad. Nauk. USSR, 51* (1946), 267–270 and *60* (1948), 29–31.

for flat plates. In both models, the cavity width and length increase without limit as $Q \downarrow 0$. These predictions also agree generally with experiment (see Section 9), for Q small. Of course, if $Q > 1$, there is usually no cavitation at all, and a well-developed cavity usually only exists if $Q < .5$; hence "cavity flows" with $Q > .5$ have no obvious physical interpretation.

It is curious that the value $Q = .5$ should affect the drag coefficient in much the same way as the Mach number $M = 1$ does. Other analogies between compressibility and cavitation (which is a form of compressibility) have been discussed by Ackeret [15] and Riabouchinsky (*Proc. Stockholm Congress Applied Mechanics* (1930), p. 156). One may note the further coincidence that, in a perfect gas, $M^2 = v^2\rho/\gamma p = 2/\gamma Q$; hence $M = 1$ does correspond roughly to $Q = .5 - 1.0$ (if $1 \leqq \gamma \leqq 2$). Finally, one may note Simmons' concept of a "blocking constant," discussed in Section 11.

A simple hydraulic (and therefore non-rigorous!) derivation of a formula closely related to (15), has been proposed by Reichardt.[22] Reichardt supposes that the energy $D\Delta s$ given off in a distance Δs by a missile, is used up in expanding the cavity radially to a volume $A^*\Delta s$. (Here D is the drag, and A^* the maximum cavity cross-section area.) Since this expansion does work against an adverse pressure difference of $p - p_c$ (cf. (12)), we infer

$$A_c = \frac{D}{(p - p_c)}, \quad \text{or} \quad \frac{A^*}{A} = \frac{C_D}{Q}, \tag{16}$$

where A is the missile cross-section.

It is encouraging that (16) agrees well with the numerical results obtained with Riabouchinsky's model, if Q is small.

One can also construct flows with $Q < 0$ having finite cavities, and otherwise defined mathematically over the entire plane. Such flows avoid the Brillouin Paradox by separating from the obstacle *behind* the point of minimum pressure, and by having cusps (cf. Fig. 12, innermost flow) at the rear end.

[22] Reichardt, H., "The laws of cavitation bubbles," *Reps. and Translations No. 766*, Ministry of Aircraft Production (Great Britain).

Flows with cusped cavities can even be expressed analytically in closed form, as shown recently by Lighthill and Allen.[23]

Southwell and Vaisey [33] have recently constructed an interesting series of plane cavity flows past a circular cylinder in a channel, with $-0.4 \leqq Q \leqq 0.0$. The cavity profiles are sketched in Figure 12. The proper interpretation of these flows is however ambiguous, for the following reasons.

First, the novel "cusped cavities" of finite length obtained theoretically, do not at all resemble physical cavities (cf. Section 9). Second, the flow obtained for $Q = 0$ is only one

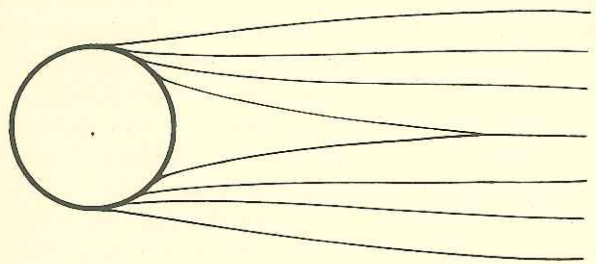

FIG. 12. Cavity flows past cylinder in channel computed by relaxation methods.

of a mathematically possible one-parameter family, and is in fact not the one usually preferred (see Section 5). Finally, in the flows with $Q > 0$, the cavity width as determined in [33] increases with Q, whereas all other evidence indicates that, in an infinite stream with $Q > 0$, the cavity width decreases as Q increases. Hence the contrary trend of Southwell and Vaisey is presumably a wall effect.[24]

These remarks are perhaps timely in view of the current interest in numerical methods. I think they illustrate the need for accompanying numerical experimentation by a profound analytical knowledge, especially as regards general existence, uniqueness, and convergence theorems.

[23] M. J. Lighthill, "A note on cusped cavities," *Aer. Res. Council Reps. and Mem. 2328* (1945), published 1949; D. N. de G. Allen, "The formation of closed wakes in fluid motions," *Quar. J. Mech. Appl. Math.*, *2* (1949), 64–71.

[24] It may also be noted that the flows of Figure 12, which are intended to be uniform to the right of a fixed line, could equally well be extended by the reflection principle to give flows like that of Figure 10.

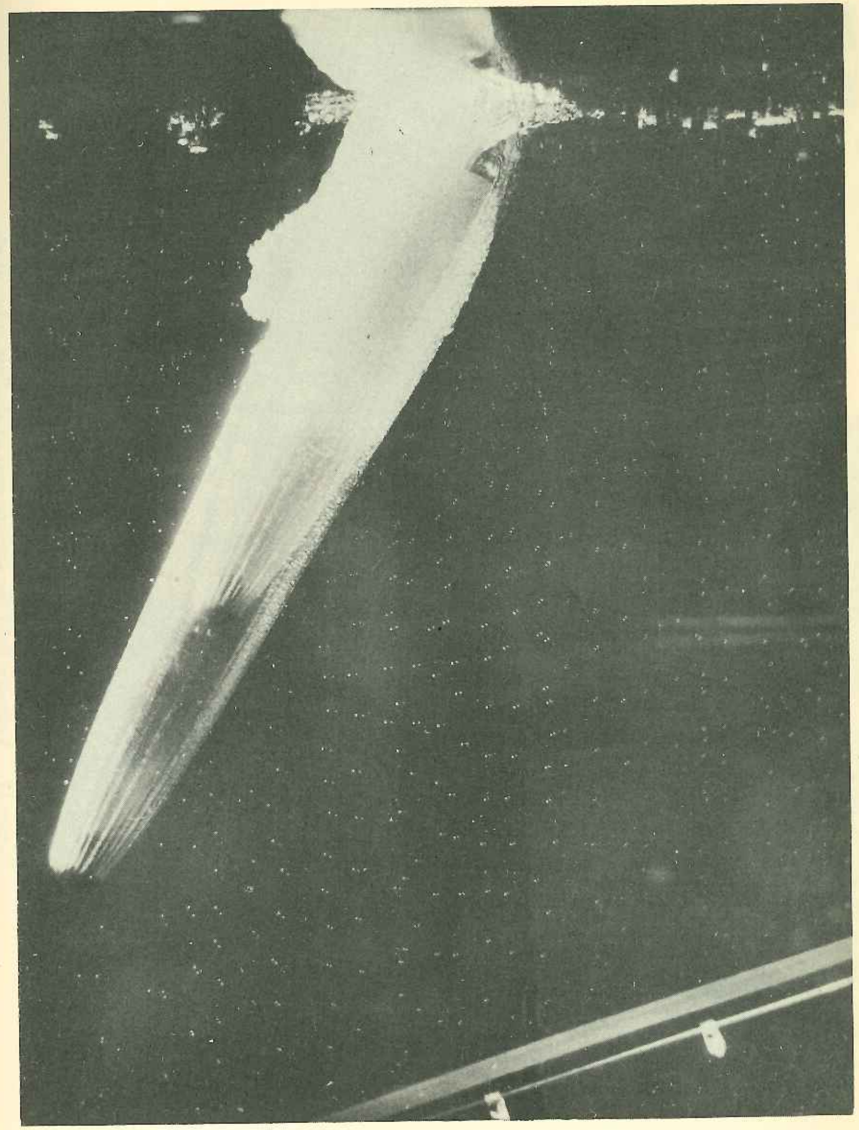

PLATE I. Velocity field and cavity behind high-speed sphere in water.

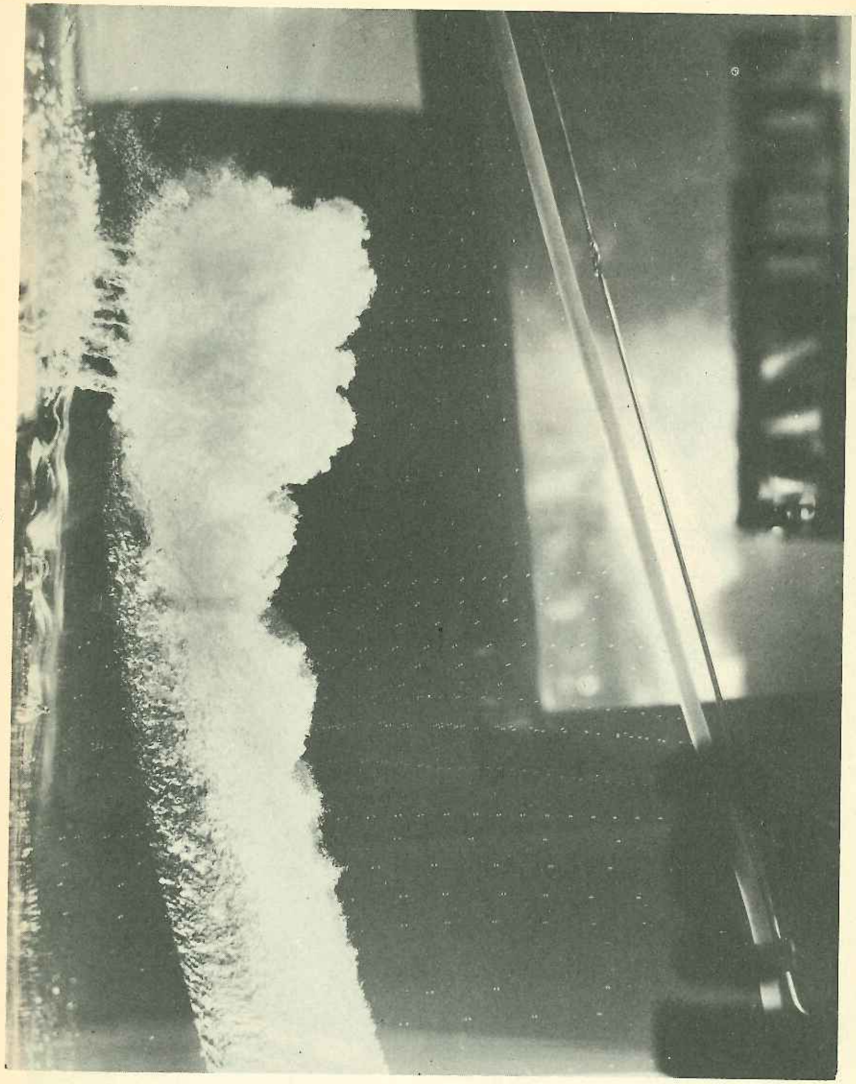

PLATE II. Velocity field for surface seal at shallow entry angle.

In this connection, the following recent fragmentary result of M. Schiffer,[25] proved by variational methods, may be mentioned. Let any body B of area β and any constant $\gamma > 0$, be given. There exists at least one plane flow around B with free streamlines, bounding a cavity of area γ. Hence there is indeterminacy as regards γ; by the Brillouin Paradox, presumably $Q < 0$. It is a plausible conjecture that, for a class of bodies as general as accolades, there is just one everywhere concave "cusped cavity," of area $\gamma(Q)$, for each $Q < 0$. It would be interesting to determine $\gamma(Q)$ in specific cases, and the limiting flow as $Q \uparrow 0$; this will presumably have an infinitely long cavity of finite width.

Although it would seem analytically difficult to construct simple flows with points of inflection on the cavity wall if $Q < 0$, no such limitation is apparent if $Q > 0$. Therefore it seems plausible that the criteria of Section 5 must be invoked, if one wishes to avoid the embarrassment of a *two*-parameter family of flows. It is an attractive conjecture that, using these criteria, there is just one symmetric flow of Riabouchinsky type past any "accolade," for sufficiently small $Q > 0$.

9. Case ρ'/ρ Near Zero: Cavity Flows

I have already mentioned the plausible hypothesis that classical "free streamline" theory is applicable when ρ'/ρ is small. Mises[26] has given elsewhere supporting data for jets into air. I shall now review recent evidence for the case of cavities behind missiles shot into water, and also for cavities formed behind obstacles held in a water tunnel with $0.2 < Q < 1.0$.

Plate I is a sample piece of evidence;[27] it shows the cavity formed by a sphere shot into water at about 150 ft/sec. Two

[25] Oral communication to the author. Using reversibility, it would seem that at least three topologically different types are possible, wake behind, "wake" in front, and symmetrically disposed double wake.

[26] R. von Mises, *Zeits. Ver. Deutsche Ing.*, *61* (1917), 447–451, 469–473, and 493–497.

[27] The technique by which it was taken is described in [19], where other similar photographs may be found.

exposures, separated by about .005 secs., are made on a single plate. The white dots speckled over the picture are small bubbles, each bubble being photographed twice. It is not hard to pair the bubbles, and the vector from the first to the second bubble of each pair is roughly proportional to the vector velocity of the water in the vicinity. In this way the cavity profile and the velocity field of the flow can be clearly visualized.

Now photographs such as these are not exactly comparable with theory, for various reasons. Thus they represent a decelerating body, and not stationary flow; the water surface presents a second free boundary, which complicates the mathematical description; the influence of air is not negligible (cf. Section 10). Nevertheless, several pertinent general correlations with theory can be made, based largely on first-hand observation.

First, except in the case of a water-tunnel with $Q > 0.3$, real cavity outlines are relatively smooth[28] and stationary for ten or more body diameters back. This agrees with the Kirchhoff model infinitely better than pictures of ordinary wakes. (See the references of Section 5.)

Second, the cavity profile is nearly always convex, and separation is in front of the cross-section of maximum diameter. This clearly supports the use of condition (iii) (hence of conditions (i)–(iv)) of Section 5, in avoiding mathematical indeterminacy.

The idea that the minimum pressure occurs in the cavity (Section 5, condition (ii)) is also supported by the observation that *vapor cavities always begin at solid surfaces* in pictures of cavity motion. Indeed, it follows immediately from Bernoulli's equation (cf. Chapter I, (6), also [9], p. 19),

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sum u_i^2 + \frac{p}{\rho} = G + \text{const.}, \tag{17}$$

for non-stationary irrotational motion of an incompressible, non-viscous fluid. In (17), $\nabla^2 G = 0$, since G is the Newtonian

[28] Small "flutings" parallel to the flow may occur unless the solid is extremely smooth; see Plate I or [35], pp. 116, 129. In turbulent flow, a rough cavity wall may occur at lower Q; see P. Eisenberg and H. Pond, *David Taylor Model Basin Report No. 668* (1948).

gravitational potential; $\nabla^2(\partial U/\partial t) = \partial(\nabla^2 U)/\partial t = 0$, (cf. Chapter I, (7)); $\nabla^2(u_i^2) = u_i\nabla^2 u_i + 2\nabla u_i \nabla u_i \geqq 0$. Hence $\nabla^2 p \leqq 0$, that is, p is *superharmonic*. It is however well-known that a superharmonic function must assume its minimum value on the boundary.[29]

Fourth, it would appear that arbitrarily long cavities (100 diameters at least), of arbitrarily great maximum cross-section, occur behind missiles which are moving sufficiently fast. This fact has a considerable practical importance, in that much of the wounding effect of high-speed bullets and bomb fragments is due to the fact that they create temporary holes much bigger than themselves.[30] For us, it has the additional significance of providing possible physical approximations to the theoretical cavities of infinite length and asymptotically infinite cross-section calculated by Kirchhoff.

Fifth, the rough experimental formula[31]

$$C_D = .55 + .4Q = .55(1 + .73Q) \tag{18}$$

for the cavity C_D of a cylinder broadside, agrees quite well with theory. Thus Brodetsky had previously derived $C_D = C_D(0) = .55$ theoretically for the "preferred" flow (cf. Section 6) at $Q = 0$, and the formula for the other values of Q follow roughly formula (15), $C_D(Q) = (1 + Q)C_D(0)$.

On the other hand, I have seen nothing[32] which at all resembles the concave "cusped cavities" with $Q < 0$ discussed above (Section 6); it would seem difficult to produce a stable wake or cavity overpressure experimentally.

[29] See G. Bouligand, *Jour. de Math.*, *6* (1927), p. 427.

[30] See [3], Section 74, and references given there; also E. N. Harvey, A. H. Whiteley, H. Grundfest, and J. H. McMillen, *The Military Surgeon*, *98* (1946), 509–528.

[31] Kempf and Foerster, *Hydromechanische Probleme des Schiffenantriebs*, Hamburg, 1932; the experimental data are due to Martyrer. For Brodetsky's result, see *Proc. Roy. Soc.*, *102* (1923), 542–553; the separation zone appears to occur experimentally somewhat nearer the equator than the 35° predicted by Brodetsky.

[32] The "deep seal" configuration of Fig. 13a is entirely transient, and it is immediately followed by a cavity with a reentrant down-jet (see [24], and [19], Figs. 21, 25).

10. *Transient Scale Effects*

The preceding results seem to confirm the plausible conjecture already stated (Section 5, end) that classical free streamline theory is applicable if ρ'/ρ is small, even though it is *not* sufficient for ν to be small. Unfortunately, recent experimental research shows that this is true, if at all, only in a very limited sense.

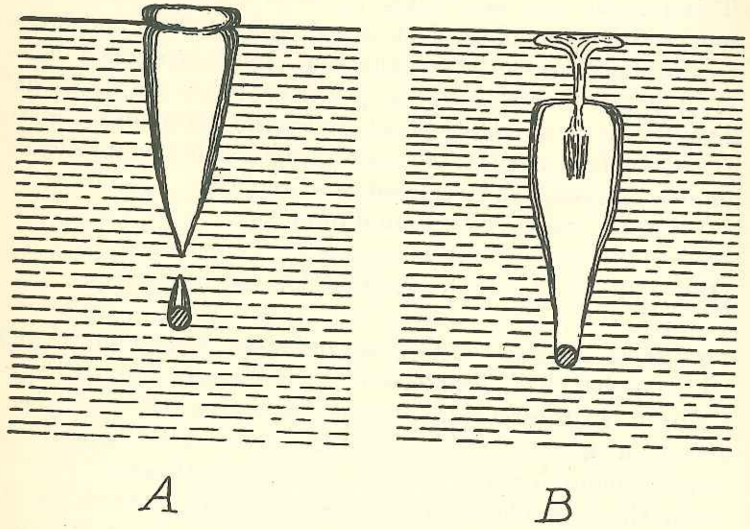

FIG. 13. Deep seal versus surface seal.

If we wish to square this conjecture with the experimental facts, we are forced to say that .0013 is not "small." Specifically, there are two hydrodynamical phenomena which occur with entry into water under atmospheric conditions, and not if the air is evacuated. Hence no mathematical theory which neglects $\rho'/\rho \leqq .0013$ can correctly predict them.

The more important is the phenomenon of *surface seal.* If a small sphere is dropped into still water at 10–20 ft/sec, the cavity first closes as in Figure 13a in a "deep seal," whereas at entry speeds of 40 ft/sec or more, the cavity first closes at the surface, as sketched in Figure 13b. A photograph of the velocity-field associated with surface seal is displayed as

Plate II. The existence of surface seal was discovered about 1900 by Worthington.[33] In 1944, R. M. Davies [23], following a suggestion of Sir Geoffrey Taylor, showed that surface seal no longer occurs if the air pressure p is sufficiently reduced.

It was at first supposed by some that whether one got surface seal or deep seal depended on the cavitation number $Q = p/\frac{1}{2}\rho v^2$, which is proportional to p and intervenes directly in other cavity phenomena. But in 1945, following my suggestion [18], Gilbarg and Anderson [24] used "heavy gases" of the Freon series, so as to vary density and pressure independently. They thereby showed that, for a given sphere diameter d and entry speed v (Froude number $F = v^2/gd$), the manner of seal was largely determined by ρ'/ρ. For example, with 1-inch diameter spheres entering at 50–150 ft/sec (i.e. $F = 10^3 - 10^4$), surface seal occurs if $\rho'/\rho > .001$; deep seal occurs $\rho'/\rho < .0001$.

This sudden change of régime near a very small value of the parameter ρ'/ρ recalls the sudden changes in ordinary wakes which occur near $1/R = \nu/vd = .02$ (alternating vortices) and $1/R = .000005$, and in pipes when $1/R = .0005$ (see Chapter I). Thus it gives us another "paradox of approximation," and illustrates again the principle that *the nature of the solutions of partial differential equations may change abruptly for extremely small values of the parameters.*

It will be shown in Chapter III, Section 16, though by a somewhat arbitrary physical analysis, that the change of régime is actually associated closely with the dimensionless parameter $N = \sqrt{F}\,\rho'/\rho$, and occurs when N is about $\frac{1}{80}$.

A second phenomenon is incompatible with the idea that problems of water motion in air can be treated as free boundary problems. Discovered about 1944 by various persons, it was studied experimentally by Prof. L. B. Slichter. Prof. Slichter showed that a smooth 2-inch diameter dural sphere, entering the water at 50 ft/sec, and at an angle of 20° with the horizontal, may be refracted downwards 5° or more when it enters. Though the exact facts are still not clear, Prof.

[33] See [35]; also W. Mallock, *Proc. Roy. Soc.*, *A95* (1918), 138–143, and E. N. Harvey and J. H. McMillen, *Jour. Appl. Phys.*, *17* (1946), 541–555.

Slichter made a careful (unfortunately unpublished) experimental analysis[34] which indicates strongly that this downward refraction is due to underpressure in the region where water is separating from the sphere—and that in turn this underpressure is caused by *air viscosity*, and its effect on the establishment of the boundary layer. However, systematic quantitative data are unfortunately still lacking.

Finally, various water tunnel studies[35] show that regardless of ρ'/ρ (which is only .00003 with water vapor cavities), the flow varies greatly as the cavitation number Q varies from zero to one. Hence, although the smallness of ρ'/ρ may be a sufficient condition for the validity of classical jet theory, it is only a useful necessary condition for the validity of cavity theory. Much work remains to be done in this connection.

11. *Wall Corrections; Stability of Pressure Coefficient*

By clarifying the physical foundations of free boundary theory, one may obtain useful and unsuspected applications to actual problems. I shall now (Sections 11–13) present some examples of this.

I shall begin with the important question of *wall effects.* Valcovici[36] showed in 1913, using the physically inept language of "wake theory," that whereas the theoretical cavity C_D ("Cavity drag coefficient") of a flat plate in an infinite stream was 0.88, as in Section 2, in a tunnel with fixed walls (as in Fig. 14a), 100 plate diameters apart it was 1.15, or 30% greater. The theoretical wall correction is reduced from 30% to less than 1% if the asymptotic "downstream" velocity v_c of the cavity wall is used instead of the usual "upstream" velocity v, in computing $C_D = 2D/\frac{1}{2}\rho v^2 A$. That is, the wall correction can be made approximately simply by computing, instead of C_D,

[34] Upward refraction occurs ordinarily; see K. Ramsauer, *Uber den Ricochetsschuss*, Thesis, Kiel, 1903; also [3], p. 453.

[35] See [15]; Kempf and Foerster, *op. cit.* supra; Eisenberg and Pond, *loc. cit.* supra; and unpublished data of Prof. R. T. Knapp.

[36] V. Valcovici, *Uber discontinuierliche Flüssigkeitsbewegungen mit zwei freien Strahlen*, Thesis, Gottingen, 1913.

$$C_D^* = \frac{D}{\frac{1}{2}\rho v_c^2 A} = \frac{D}{(p_s - p_c)A}, \tag{19}$$

by Bernoulli's Theorem. (Here p_s is the stagnation pressure, p_c the cavity pressure, and A the model cross-section.)

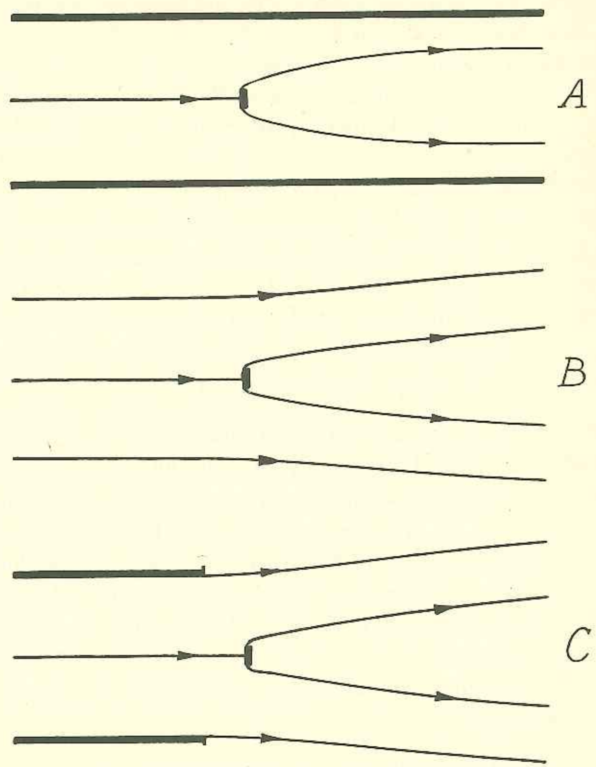

FIG. 14. Flows past plate in closed channel, free jet, and bounded jet.

Valcovici also showed that the wall correction became negligible if a jet with free boundaries (Figure 14b) was used instead of a stream between fixed walls. Recently, Milton Plesset and I [20] have noted that Valcovici's formulas could be derived easily by the method of Section 4, since the hodograph of half the flow is a quarter-circle; we also obtained numerical results for bounded jets (Figure 14c). Similar results were obtained by N. Simmons for a class of flows with $Q > 0$ including those treated by Riabouchinsky.

Now it is almost certain that the original physical interpretation of Valcovici's result as a wall correction for wake drag is entirely erroneous. And it is only honest to say that my engineering friends have been very skeptical of the interpretation as a wall correction for cavity drag. They point out that the theory neglects the pressure gradient of the "undisturbed stream." This is another wall effect caused by eddy viscosity, which is well-known in hydraulics—if not in classical hydrodynamics.

Nevertheless, I maintain that formula (19) is both mathematically and physically sound. In fact, I should like to assert the validity of the following stronger *principle of stability of the pressure coefficient:* for an obstacle of given shape in a water tunnel (or jet!), *the pressure profile function*

$$\frac{(p - p_c)}{(p_s - p_c)} = \frac{(p - p_c)}{\frac{1}{2}\rho v_c^2} \tag{19'}$$

is insensitive to the presence of walls and changes in Q.

Thus it was proved rigorously in [20] that (19′) is true for wedges of any angle symmetrically placed in a water tunnel. Again, formula (15) for the variation of C_D with Q, which was obtained by Gilbarg by study of the numerical evidence—and which has received rough experimental confirmation (see (18))—is equivalent to (19′). To see this, it suffices to note that by Bernoulli's theorem $Q = \frac{1}{2}\rho(v_c^2 - v^{*2})/\frac{1}{2}\rho v^{*2}$, where v^* is the (up-stream) free stream velocity. Hence, by simple algebra,

$$Q = (v_c/v^*)^2 - 1, \qquad 1 + Q = v_c^2/v^{*2},$$

and

$$C_D = \frac{2D}{\rho v^{*2} A} = \left(\frac{v_c^2}{v^{*2}}\right) C_D^* = (1 + Q)C_D^*. \tag{19''}$$

Finally, I note that Ackeret [15] obtained consistent results by assuming (19′), in extrapolating from his observations for $Q > 0.25$, to the case $Q = 0$.

In fact, it seems to me that the "principle of stability of the pressure coefficient" has all the earmarks of a successful

hydraulic principle—including an inability to stand too close scrutiny.

Even if it is inconvenient to measure $p_s - p_c$ (which must however be known if Q is to be measured), one can make an estimate of the correction in (19). If T is the tunnel cross-section, and A^* the maximum cavity area, then the mean velocity at that point is $vT/(T - A^*)$. Since the cavity velocity v_c is (the cavity being convex) at least the mean velocity, we infer

$$C_D^* \geqq \frac{C_D}{(1 - A^*/T)^2}. \tag{20}$$

Equality holds in Valcovici's case, and more generally if there is a downstream asymptotic cavity velocity and maximum cross-section.

Formula (20) reveals area ratios for space flows as the logical analogs of diameter ratios for plane flows. This suggests that a 30% wall correction will apply as $Q \downarrow 0$ for cavity flows past a flat disc in a tunnel ten times as wide.

Comparison of (20) with (19″) also reveals the analogy between wall corrections and corrections for Q. Both (cf. (16)) effectively limit the indefinite growth of cavities. In fact (cf. Section 6, before footnote one), it is quite possible to identify the two, setting $Q = (v_c/v^*)^2 - 1$. Similarly, every water tunnel with fixed walls has a *blocking constant;* one can prove for plane or axially symmetric flow that (cf. [20])

$$Q \geqq 2\sqrt{C_D \frac{A}{T}} \geqq \sqrt{\frac{A}{2T}} \qquad \text{if} \qquad C_D \geqq .0625. \tag{20'}$$

Hence one cannot get very low Q in such tunnels: thus to achieve $Q = 0.05$, we must have $T/A \geqq 400$.

12. Lined Cavity Charges

So far I have mentioned almost no recent progress involving jets. However, a very important application of the classical theory of *impinging jets* was discovered in the late war, to *lined cavity charges*, such as were used in the American "bazooka," the British P.I.A.T., and many other anti-tank and demolition weapons. I shall briefly summarize this application of free streamline theory; a complete account may be found in [21].

The construction and performance of such weapons may be described in principle as follows. An explosive charge, surrounding a hollow "cavity" lined with metal, is detonated from the rear. I shall consider only the cases of conical and wedge-shaped liners; a sectional view of these is given in Figure 15a. As the charge explodes, it drives the liner inward and forward, and it is found that the liner so impelled has an enormous penetrating power. How can we explain this power?

The best known explanation proceeds from the following plausible approximate assumptions. Assumption 1: After

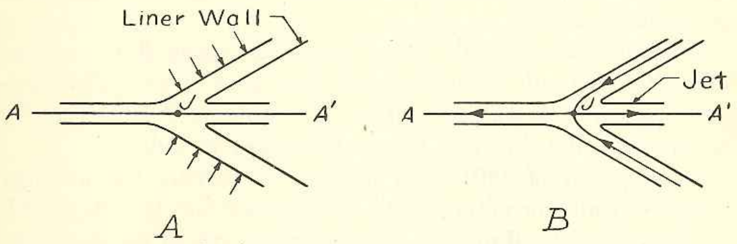

FIG. 15. Collapsing cone and impinging jets.

receiving an initial impulse from the explosive, the liner walls move inward under their own momentum with constant velocity, until they meet on the "axis" AA' of Figure 15a. Assumption 2: Under the terrific stresses generated, the liner behaves like a perfect fluid. Assumption 3: Relative to axes moving with the junction J of opposite walls, the flow is stationary. Assumption 4: The surfaces of the liner walls are free boundaries.

These assumptions reduce the problem to the classical case of "impinging jets" (see Figure 15b). It is a matter of high-school algebra and trigonometry, combined with the laws of conservation of mass and momentum, to predict the velocities of the jet JA' and residual slug AJ formed by the metal liner, and to predict the ratio of their masses. Especially, the extremely high predicted velocity of the jet is important, as explaining the effectiveness of lined cavity charges as anti-

tank weapons. The mathematics of penetration of armor by high-speed jets will be described in Section 13.

But first I should like to call attention to the peculiar fact that, though the advanced mathematical theory of impinging plane jets may be found in various places,[37] I know of no previous application of this theory, or discussion of the axially symmetric analog. Also, it seems significant to me that the practical usefulness of the mathematical model does not come from the relatively advanced use of complex variable theory and conformal mapping (the "hodograph" is a circle, and the method of Section 4 is applicable in principle). Instead, it is humble mathematical methods, combined with simple physical conservation principles, which are really useful. This is especially true because, as in the determination of the coefficient of contraction of the Borda tube (Section 3), it is only the simple laws of conservation which are applicable to the axially symmetric case of greatest practical importance.

13. *Penetration by Liquid Jets*

It has been known since Worthington's early photographs [35], that cavity "deep seal" (Figure 13a) is followed by the formation of upward jets. Gilbarg and Anderson [24] have recently shown that *down-jets* also occur, and the cavity flows with reentrant jets described in Section 6 may be regarded as crude approximations to the flow induced by such down jets.[38] These jets are not to be confused with those arising from lined hollow charges (Section 12).

A mathematical model for stationary plane *up-jets* has recently been suggested by T. E. Caywood and myself [19]; it is surprising that it should have been discussed so slightly in the classical literature.[39] The so-called "physical" z-plane

[37] See [10], p. 280; [25], pp. 11–12; W. Schach, *Ing. Archiv*, *5* (1934), 245–265. In the plane, half the flow corresponds to a jet impinging on a plate, which is easily realized physically. One gets the impression from the figure on p. 381 of the 4th edition of Gibson's *Hydraulics*, that even the usual topological picture is not realized physically in the case of impinging jets.

[38] Photographs displaying the velocity-fields of up-jets and down-jets, like those of Plates I–II, may be found in [19].

[39] I was only able to find, in [25], one passing reference.

is shown in Figure 16, with the up-jet along the x-axis SS'. The "hodograph" of half the flow is a semi-circle, while the W-domain is a half-plane with semi-infinite slit SOS'. Hence the method sketched in Section 4 is applicable; the details follow. By the elementary theory of Schwarz-Christoffel transformations, the W-domain can be mapped onto the upper half t-plane by a transformation

$$(21) \qquad dW = \left(1 - \frac{1}{t}\right) dt, \qquad \text{or} \qquad W = t - \ln t.$$

The most general transformation of the hodograph is given as before by

$$(21') \qquad t = \frac{A\sigma + B}{C\sigma + D} = c\,\frac{\zeta^2 + 2\lambda\zeta + 1}{\zeta^2 + 2\mu\zeta + 1}, \qquad \text{where}$$

$$\sigma = \frac{1}{2}(\zeta + \zeta^{-1}).$$

But by Figure 16, $\zeta = -1$ when $t = \pm\infty$ at S' and T; $\zeta = 0$ when $t = 1$ at 0; and $\zeta = 1$ when $t = 0$ at S. Hence $\lambda = -1$, $\mu = 1$, and by suitable choice of units we can make

$$(21'') \qquad t = \frac{(1-\zeta)^2}{(1+\zeta)^2}.$$

The preceding model may easily be altered to describe penetration of an incompressible fluid by a two-dimensional jet of different density, as of tank armor by the jet from a wedge-shaped liner. The "dividing streamline" TOT' of Figure 16 may be taken as a streamline of discontinuity (*not* a free streamline, however!). Above TOT' is the penetrating jet of density ρ moving *downwards* with speed v, obtained from the model of the preceding paragraph by multiplying all vector velocities above TOT' by the scalar $-v$. Below TOT' is the material being penetrated, of density ρ'; we assume that it behaves like a liquid; and obtain its velocity field by multiplying all velocities below TOT' in the preceding paragraph by the scalar $-\sqrt{\rho/\rho'}\,v$. By Bernoulli's equation, the pressure is continuous across the streamline of discontinuity.

Thus we have a *stationary streamline of discontinuity not a free streamline;* this type of flow seems to be new.[40] By superposing a constant vertical velocity on the whole, we get a model for the penetration of a stationary target by a liquid jet.

What about axially symmetric jets? Again, the important physical quantity, the *rate* of penetration, can be obtained by use of algebra and energy conservation (disguised as the

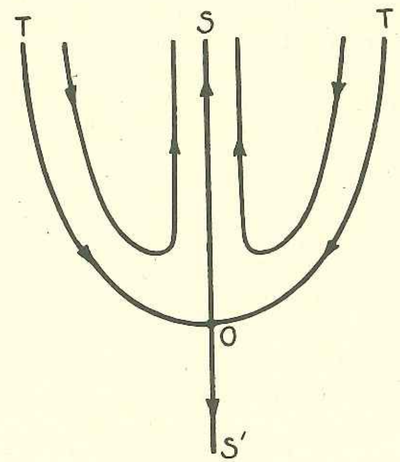

FIG. 16. Collapsing cavity—penetration by jet.

Bernoulli equation) alone, without the benefit of higher mathematics! The details are mathematically so trivial that I shall not bore you with them (they may be found in [21]).

14. *Stationary Cavities in Accelerated Motion*

A novel type of non-stationary plane motion with stationary free boundaries has been found very recently by von Karman.[41] Consider the Bernoulli equation (17) of Section 9, with

[40] See [17], where flows of this type are discussed in detail. Continuous analogs of such changes of density have been introduced by M. Munk and R. Prim, *Proc. Nat. Acad. Sci. U.S.A.*, *33* (1947), 137–141.

[41] Th. von Karman, "Les sillages non stationnaires"; Prof. von Karman kindly lent me a copy of this unpublished note. Karman's use of similarity is analogous to that of Wagner, described in Chapter IV, Section 8.

gravity neglected,

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sum u_i^2 + \frac{p}{\rho} = C(t), \tag{22}$$

for non-viscous fluid. We look for flows involving accelerated motion past a fixed obstacle, in which dependence on time and space can be separated, so that

$$U = f(t)\phi(x_1,x_2,x_3), \qquad \text{and} \qquad p = g(t)\psi(x_1,x_2,x_3). \tag{23}$$

If we substitute from (23) in (22), we get

$$f'\phi + \frac{1}{2}f^2(\nabla\phi\nabla\phi) + \frac{g\psi}{\rho} = C(t). \tag{22'}$$

For this to be an identity, since ϕ, $\nabla\phi\nabla\phi$, ψ/ρ, are functions of $\mathbf{x}$ alone, clearly f' and g must bear a constant ratio to f^2, and ϕ and ψ a constant ratio to $\nabla\phi\nabla\phi$.

Setting $f' = -kf^2$, we get $-kdt = df/f^2$, or (by appropriate choice of time-origin) $f = 1/kt$. This is of course the case of constant *acceleration coefficient*, and not of constant acceleration.

Now a *free* streamline is one with $\psi =$ const.; hence we have as a necessary and sufficient condition

$$-k\phi + \tfrac{1}{2}(\nabla\phi\nabla\phi) = C_1, \tag{24}$$

or

$$U = \frac{v^2}{2k} + C^*/t. \tag{25}$$

In the case $k = 0$ of no acceleration, (24) reduces of course to the classic condition (Section 1), that velocity is constant on a free streamline with stationary motion. Differentiating (25) along the streamline, since $v = dU/ds$, we get (indicating differentiation with respect to s by an accent), $-kU' + U'U'' = 0$. Except at stagnation points where $U' = 0$, this amounts to $U'' = k$, or $v = U' = ks + C$: *velocity is a linear function of distance along the streamline.* Choosing constants of integration suitably, we easily reduce to

$$U' = ks, \qquad U = \tfrac{1}{2}ks^2 + C^*/t.$$

Using (25), one can apply a hodograph method, and von Karman has by this means obtained explicitly an accelerated

plane cavity flow past a flat plate held broadside. Curiously, a solution is possible in the large only for one value of k!

The preceding refers to a fixed plate in a fluid accelerated at infinity. We can transform to the case of an accelerated plate in a fixed fluid, by superposing a "virtual" gravity field, constant in space at any instant. Now if the cavity is filled with fluid of density ρ, such a field—or any other conservative field—has no effect on the flow (Chapter III, Section 12). Hence von Karman's model describes a wake behind an accelerated or decelerated object, for the case $\rho'/\rho = 1$ and constant acceleration or deceleration coefficient such as occurs with constant C_D.

It is perhaps of interest to obtain an analog of (25) for streamlines of discontinuity not free streamlines. Along any fixed streamline, let $\mathbf{u} = f(t)\mathbf{v}(\mathbf{x})$, where $f(t) = 1/kt$. The component of the Equations of Motion parallel to the streamline reduces to

$$\partial u/\partial t + udu/ds + dp/\rho ds = 0,$$

where u is the scalar speed of flow, and s measures distance along the streamline. Substituting for $u = v/kt$, we get $\partial u/\partial t = -u/t$, whence we have the generalized Bernoulli equation

$$\frac{1}{2}u^2 + \frac{p}{\rho} = \frac{1}{t}\int uds, \quad \text{or} \quad \frac{1}{2}v^2 + \frac{k^2t^2p}{\rho} = k\int vds \tag{26}$$

in the case of a fluid of constant density ρ. Since p is continuous across any streamline of discontinuity,

$$\rho\left[\frac{1}{2}u^2 - \frac{1}{t}\int uds\right] = \rho_1\left[\frac{1}{2}u_1^2 - \frac{1}{t}\int u_1ds\right], \tag{26'}$$

where subscripts refer to opposite sides of the streamline.

15. *Axially Symmetric Flows*

Until the past few years, the only significant non-planar flows with free streamlines which had been computed, were the (axially symmetric) jet through a circular orifice in a flat plate, and the impact of a circular jet on a flat plate. The former was calculated by Trefftz,[42] using successive approximations based on integral equations; he computed the coefficient of contraction to be 0.68. The latter was calculated by Schach, using the same method.

Recently, Southwell and Vaisey [33'] have shown the power of "relaxation methods," by calculating the jet through a circular orifice, from a container of different dimensions, and computing besides the axially symmetric jet through a Borda tube in the shape of a circular cylinder. In both cases the coefficient of contraction was assumed—as "generally accepted" to be 0.68 in Trefftz' case, and as 0.50 in the Borda case (we showed this in Section 3). In addition they computed a flow with a "cusped cavity" past a sphere.

These achievements are obviously most promising, and it is to be hoped that "relaxation methods" may also obtain flows past a sphere with $Q \geqq 0$, as these seem physically more interesting.[43] Also, it is to be hoped that the ambiguities specified above (Section 8) can be eliminated. Indeed, a rigorous justification of either the Trefftz or Southwell-Vaisey method would be most valuable, as would an accurate comparison of their numerical results.

In the same connection, one should mention the very recent use of the electrolytic tank analogy to obtain the axially symmetric Riabouchinsky flow past a disc in a channel.[44] When (15) was used to extrapolate the results to $Q = 0$, the estimate $C_D(0) = 0.823$ was obtained, which agrees very well with experiment.

Another impressive recent achievement is the derivation by Levinson [30] of analogs of formulas (11)–(11′), for the

[42] E. Trefftz, *Uber die Kontraktion kreisformiger Flussigkeits strahlen*, Dissertation, Strassburg, 1914; also *Zeits. Math. u. Phys.*, *64* (1916), p. 34. Schach's results are in the *Ingenieur Archiv*, *6* (1935), 51–58. Note also Weingartner, *Gott. Nachr.* (1890), and H. Reissner, *Zeits. ang. Math. Mech.*, *12* (1935), 25–35.

[43] It would seem difficult to estimate "wall effects" by relaxation methods; so many mesh points would be needed to get any accuracy; cf. Section 11.

[44] "Détermination des lignes de jet dans les mouvements plans et de révolution," *Problème D359A*, *Travaux du Laboratoire de Calcul Expérimental Analogique*, by P. Marchet, under the direction of L. Malavard.

axially symmetric case. Assuming only that for some s with $0 < s < 1$, the cavity profile can be expressed by

$$(27) \qquad y = x^s g(x), \quad \text{where} \quad \lim_{x\to\infty} \frac{xg'(x)}{g(x)} = 0,$$

Levinson proved that the cavity profile must satisfy

$$(28) \qquad y = \sqrt{Cx}\, g(x), \quad \text{where} \quad (\ln x)^{-\frac14-\epsilon} < g(x) < (\ln x)^{-\frac14+\epsilon}$$

for all $\epsilon > 0$ and sufficiently large x. He proved further that if (18) is strengthened to $xg'(x)/g(x) = 0(1/\ln x)$, then

$$(28') \qquad y \sim \sqrt{Cx}\,(\ln x)^{-\frac14} \qquad \left(\text{i.e., } \left\{\lim_{x\to\infty} \frac{y(\ln x)^{\frac14}}{\sqrt{Cx}}\right\} = 1\right),$$

and the drag is given by

$$(28'') \qquad D = \frac{\pi\rho C^2 v^2}{8}.$$

However, Levinson failed to prove the existence of a solution. In view of the complexity of (28′), and of the Paradoxes of Overdeterminacy which are known (cf. Chapter I), one may well wonder if there exist axially symmetric cavity (alias "wake") flows, in the classical mathematical sense.

A similar but much easier result has also been recently obtained for plane and axially symmetric cavity flows with reentrant jets ([6], Figure 10). I have shown [17] that the jet cross-section A^* and the cavitation number Q are related by

$$(29) \qquad C_D = \frac{2(1 + Q + \sqrt{1+Q})A^*}{A} \quad \text{whence} \quad \frac{A^*}{A} = \frac{C_D}{4} \quad \text{if} \quad Q = 0.$$

The asymptotic considerations needed to prove (20) are quite like those used in deriving the d'Alembert paradox and Levi-Civita's formula (11′).

16. *Summary of Recent Tendencies*

In closing this chapter, I should like to stress the *trend* of recent progress in free boundary theory. I think it is clear that there have been important developments dealing with an *increasing variety of mathematical problems, inspired by an awareness of the great variety of real physical problems.*

Thus the trend of recent progress has been in the same general direction as modern theoretical physics. Twentieth-century physics has also been characterized by a broadening of foundations, both in quantum mechanics and relativity, and by an acceptance of the philosophy that we are not yet able to deduce all the laws of Nature from a single set of convenient and harmonious mathematical hypotheses.

I hope that a realization of this community of spirit between modern physics and fluid mechanics will lead to a greater interest of physicists generally in the theoretical and experimental problems of fluid mechanics. It seems surprising that this subject, to which so much was contributed by Stokes, Helmholtz, Rayleigh, and other physicists of the nineteenth century, should owe all its more recent progress to mathematicians and engineers. I feel confident that a renewed interest by physicists in the accurate correlation of theory with experiment, would help greatly in providing it with a more solid scientific basis.

CHAPTER III

Modeling and Dimensional Analysis

1. Introduction

The use of *models* to study fluid mechanics has an appeal for everyone endowed with natural curiosity. What active boy has not played with ship and airplane models, or crude models of dam and drainage systems? Even in the most advanced technical engineering, such models play a fundamental and indispensable role.

And yet in few departments of the physical sciences is there a wider gap between theory and practice, between scientific knowledge and the state of the art, than in the use of models to study hydrodynamic phenomena. This is partly because engineers with adequate laboratory facilities, who are most familiar with the realities of the situation, are usually preoccupied by special practical problems concerning one type of model. Whereas the academic scientist, preoccupied by the difficulties of logical exposition, too often ignores experimental reality.

It is my aim to narrow this gap; this can be done only by a critical examination of the whole field from both sides.

Being a mathematician, I shall begin with the *theory* of modeling. Here there are a number of simple, yet fundamental, mathematical remarks which have hitherto escaped general attention—no doubt because mathematicians have not seriously studied the subject. I hope I can make my remarks in sufficiently plain language, so that physicists and engineers will see their relevance.

2. Dimensional Analysis

A priori theoretical arguments for the use of models usually follow one of two main lines: so-called *dimensional analysis*,