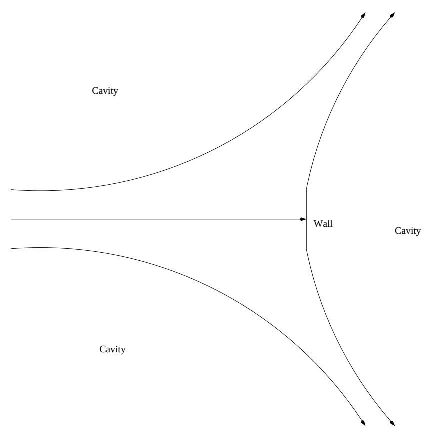
MA 342H
Assignment 4
Due never

`Id: 342H-s2014-4.m4,v 1.3 2014/04/26 18:24:58 john Exp john`

1. An inviscid, incompressible, irrotational, steady planar flow has a free stream which hits a (finite) wall and splits into two free streams, as in the diagram. The directions of the outgoing streams form an angle $\pm\varphi$ with that of the incoming stream. Find the free streamlines.

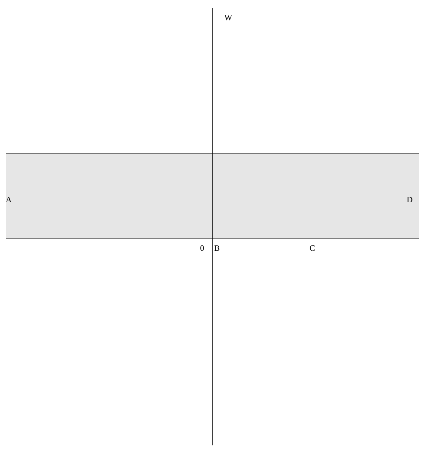


   Note: For simplicity, assume units have been chosen such that the free stream speed is 1, the density of the fluid is 1 and the rate of flow is 2π. Choose the origin of the coordinate system at the centre of the wall and assume the stream function and velocity potential are zero there. Orient the coordinate system so that the wall runs along the y-axis.
   Hint: Use the complex hodograph method. The flow is symmetric about the x-axis, so you need only consider the part of the flow in the upper half-plane.

Solution: As suggested, we consider only the part of the flow in the upper half-plane. This is bounded by a pair of streamlines. One of these follows the negative x-axis, then follows the wall up until it ends, and then follows a free streamline. The other boundary is a free streamline everywhere. On each of these streamlines the stream function is constant. Since it was set equal to zero at the origin it must be zero everywhere on the first streamline. The rate of flow between these two streamlines is one half the total flow, *i.e.* π, so the stream function is equal to π on the second streamline, and the allowed values of the complex potential are

$$0 < \operatorname{Im} W < \pi.$$

This is shown in the figure below.

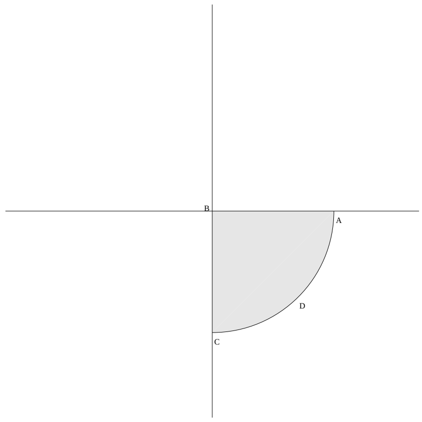


The labeling of the points will be explained later. Points at infinity are, of course, shown as far from the origin as possible in the diagram.

Inside the fluid $|\zeta| < s = 1$ everywhere. From the picture we expect that the both the horizontal and vertical components of velocity in the upper half-plane are positive, so the allowed valued of the complex conjugate velocity are

$$|\zeta| < 1, \quad \operatorname{Re} \zeta > 0, \quad \operatorname{Im} \zeta < 0.$$

This is shown in the figure below.



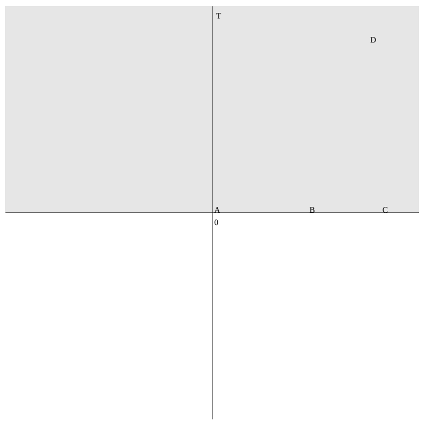
Just as in the example of the flow through a hole, done in lecture, it is easier to work with

$$T = \exp(W)$$

in place of W. The allowed values of T are just the upper half-plane

$$\operatorname{Im} T > 0.$$

This is shown in the figure below.



If we can find the relation between T and ζ then we can recover z by integrating

$$dz = \frac{dW}{\zeta} = \frac{dT}{\zeta T}.$$

To determine the relation between T and ζ we need to consider not only the allowed regions but also some points on the boundary, corresponding to physically important points of the flow. We consider four such points, which will be called A, B, C, and D. A is infinitely far upstream, where $\zeta = 1$ and $U = \operatorname{Re} W = -\infty$. There $T = 0$. B is the stagnation point at the centre of the wall, where $\zeta = 0$ and $W = 0$. There $T = 1$. C is the upper tip of the wall. There $\zeta = -i$ and $W = \alpha$, where α is the value of the velocity potential at the tip of the wall, which is unknown. Here $T = e^\alpha$. Finally, D is a point infinitely far downstream, where $\zeta = e^{-i\varphi}$ and $U = \operatorname{Re} W = +\infty$. Here $T = \infty$.

Considering T as a function of ζ, the reflection principle gives us four relations which this function must satisfy, although two of these turn out to be the same. From the segment AB we get

$$\overline{T(\zeta)} = T(\overline{\zeta}).$$

From the segment BC we get

$$\overline{T(\zeta)} = T(-\overline{\zeta}).$$

Finally, both the segments CD and DA give

$$\overline{T(\zeta)} = T\left(\frac{1}{\overline{\zeta}}\right).$$

Combining these in various ways we find

$$T(\zeta) = T(-\zeta) = T(\zeta^{-1}) = T(-\zeta^{-1}).$$

It follows that T depends on ζ only through $\zeta^2+\zeta^{-2}$. Comparing values at the points A, B, C and D we are led to

$$T(\zeta) = \frac{\zeta^2 - 2 + \zeta^{-2}}{\zeta^2 - 2\cos(2\varphi) + \zeta^{-2}}$$

and also a value for the constant α which appeared earlier

$$\alpha = \log\left(\frac{2}{1+\cos(2\varphi)}\right),$$

but we won't need this.

Differentiating,

$$dz = \frac{dT}{\zeta T} = \frac{2}{\zeta^2}\left(\frac{\zeta+\zeta^{-1}}{\zeta-\zeta^{-1}} - \frac{\zeta^2-\zeta^{-2}}{\zeta^2 - 2\cos(2\varphi)+\zeta^{-2}}\right) d\zeta$$

or, after a bit of partial fractions,

$$dz = 2\frac{d\zeta}{\zeta-1} - 2\frac{d\zeta}{\zeta+1} + e^{-i\varphi}\frac{d\zeta}{\zeta-e^{i\varphi}} + e^{i\varphi}\frac{d\zeta}{\zeta-e^{-i\varphi}} - e^{-i\varphi}\frac{d\zeta}{\zeta+e^{i\varphi}} - e^{i\varphi}\frac{d\zeta}{\zeta+e^{-i\varphi}}.$$

Integrating

$$z = 2\log\left(\frac{1-\zeta}{1+\zeta}\right) + e^{-i\varphi}\log\left(\frac{1-\zeta e^{-i\varphi}}{1+\zeta e^{-i\varphi}}\right) + e^{i\varphi}\log\left(\frac{1-\zeta e^{i\varphi}}{1+\zeta e^{i\varphi}}\right).$$

The constants of integration have been chosen such that $z = 0$ when $\zeta = 0$. The boundaries of the flow are given by substituting $\zeta = e^{-i\theta}$, for $0 < \theta < \frac{\pi}{2}$:

$$z = 2\log\left(\frac{1-e^{-i\theta}}{1+e^{-i\theta}}\right) + e^{-i\varphi}\log\left(\frac{1-e^{-i\varphi-i\theta}}{1+e^{-i\varphi-i\theta}}\right) + e^{i\varphi}\log\left(\frac{1-e^{i\varphi-i\theta}}{1+e^{i\varphi-i\theta}}\right).$$

We could stop here, but if we want to trace the free streamlines accurately then it is worth doing a bit more work:

$$\begin{aligned} z &= 2\log\left(i\tan\frac{\theta}{2}\right) + e^{-i\varphi}\log\left(i\tan\frac{\theta+\varphi}{2}\right) + e^{i\varphi}\log\left(i\tan\frac{\theta-\varphi}{2}\right) \\ &= 2\log\left(\tan\frac{\theta}{2}\right) + e^{-i\varphi}\log\left(\tan\frac{\theta+\varphi}{2}\right) + e^{i\varphi}\log\left(\tan\frac{\theta-\varphi}{2}\right) \\ &\quad + \pi i\left(1 + \frac{1}{2}e^{-i\varphi} \pm \frac{1}{2}e^{i\varphi}\right), \end{aligned}$$

where the sign is positive if $\varphi < \theta$ and negative if $\varphi > \theta$. We see in this way, for example, that the upper edge of the wall, where $\theta = \frac{\pi}{2}$, is located at the point

$$\left[\pi + 2\log\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\sin\varphi + \cos\varphi\right] i.$$