# MA 342H
# Assignment 3
# Due 2 April

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1. The usual way to solve the heat equation

$$u_t - \kappa u_{xx} = 0$$

numerically is to approximate the $t$ derivative with a forward difference and $x$ derivatives with centred differences:[1]

$$\frac{U(t+k,x) - U(t,x)}{k} - \kappa \frac{U(t,x+h) - 2U(t,x) + U(t,x-h)}{h^2} = 0.$$

As with the finite difference scheme for the wave equation discussed in class, this scheme may or may not exhibit numerical instability, depending on the values of $\kappa$, $h$ and $k$. Determine, using the same method used in class for the wave equation, where the stability threshold lies.
*Solution:* As usual, we write $U_{m,n} = U(mk, nh)$, so that

$$\frac{U_{m+1,n} - U_{m,n}}{k} - \kappa \frac{U_{m,n+1} - 2U_{m,n} + U_{m,n-1}}{h^2} = 0$$

or

$$U_{m+1,n} = pU_{m,n+1} + (1-2p)U_{m,n} + pU_{m,n-1}$$

where $p = \kappa k/h^2$. To determine stability we look at solutions of the form

$$U_{m,n} = c_m e^{2\pi i n\xi}.$$

[1] We write $U$ in place of $u$ to distinguish the approximate solution from the exact solution.

The coefficients $c_m$ satisfy

$$c_{m+1} = \left(1 - 4p\sin^2(\pi\xi)\right) c_m.$$

If $p \leq 1/2$ then solutions stay bounded. Indeed they decay exponentially if $p < 1$ and $\xi \notin \mathbf{Z}$. If $p > 1/2$ then there is a range of values of $\xi$, including $\xi = 1/2$, for which solutions grow exponentially. The stability threshold is therefore $p = 1/2$.

2. The differential equation

$$y_{xx} = \alpha y^2 + \beta y$$

in an interval $[a, b]$ is the Euler-Lagrange equation for the first order Lagrangian

$$L(x, y, y_x) = \frac{1}{2}y_x^2 + \frac{\alpha}{3}y^3 + \frac{\beta}{2}y^2.$$

The equation can be solved exactly in terms of elliptic functions, but one can also solve it numerically, either by finite differences or finite elements. Derive the equations for a finite element scheme with continuous piecewise linear elements.

*Solution:* We split $[a, b]$ into $n$ equal subintervals $I_0 = [x_0, x_1]$, ..., $I_{n-1} = [x_{n-1}, x_n]$, where $x_j = a + jh$ and $h = (b - a)/n$. A continuous piecewise linear function is determined by its values at the endpoints of these subintervals:

$$y(x) = \frac{y(x_j)(x_{j+1} - x) + y(x_{j+1})(x - x_j}{h}$$

for $x \in I_j$. Then

$$y_x(x) = \frac{y_{j+1} - y_j}{h}$$

in the interior of $I_j$. In each subinterval $L(x, y(x), y_x(x))$ is a polynomial of degree at most 3 in $x$, so, by Simpson's rule,

$$\int_{I_j} L(x, y(x), y_x(x))\, dx = \frac{L_j + 4L_{j+1/2} + L_{j+1}}{6}h$$

where

$$\begin{aligned} L_j &= L\left(x_j, y(x_j), y_x(x_j)\right) \\ L_{j+1/2} &= L\left(x_{j+1/2}, y(x_{j+1/2}), y_x(x_{j+1/2})\right) \\ L_{j+1} &= L\left(x_{j+1}, y(x_{j+1}), y_x(x_{j+1})\right) \end{aligned}$$

and

$$x_{j+1/2} = \frac{x_j + x_{j+1}}{2}$$

is the midpoint of $I_j$. Substituting

$$\begin{aligned}\int_{I_j} L(x, y(x), y_x(x))\, dx = {} & \frac{1}{2h}\left(y_j^2 - 2y_j y_{j+1} + y_{j+1}^2\right) \\ & + \frac{\alpha h}{6}\left(y_j^2 + y_j y_{j+1} + y_{j+1}^2\right) \\ & + \frac{\beta h}{12}\left(y_j^3 + y_j^2 y_{j+1} + y_j^2 y_{j+1} + y_{j+1}^3\right).\end{aligned}$$

To simplify this and later equations we use the notation $y_j = y(y_j)$. Then

$$\int_a^b L(x, y(x), y_x(x))\, dx = \sum_{j=0}^{n-1} \int_{I_j} L(x, y(x), y_x(x))\, dx.$$

For a stationary point we need

$$0 = \frac{\partial}{\partial x_k} \int_a^b L(x, y(x), y_x(x))\, dx = \frac{\partial}{\partial x_k} \sum_{j=0}^{n-1} \int_{I_j} L(x, y(x), y_x(x))\, dx.$$

Only the $j = k - 1$ and $j = k$ summands contribute. After multiplying by $12h$ and grouping terms,

$$\begin{aligned}(12 - 2\alpha h^2) y_{k+1} - (24 + 8\alpha h^2) y_k + (12 - 2\alpha h^2) y_{k-1} \\ = \beta h^2 \left(y_{k+1}^2 + 2y_k y_{k+1} + 6y_k^2 + 2y_k y_{k-1} + y_{k-1}^2\right).\end{aligned}$$