MA 342H
Assignment 2
Due 5 March 2014

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Id:  342H-s2014-2.m4,v 1.2 2014/04/01 23:12:19 john Exp john
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1. Solve the fully nonlinear first order scalar equation

$$(1 + x^2 + y^2)^2(u_x^2 + u_y^2) = 1$$

with initial conditions

$$u(x, 0) = 0.$$

There are two solutions. Take the one where $u_y(x, 0) > 0$.
Note: The characteristic equations can be solved explicitly, and then the variables s and t can be eliminated, but this is far from easy. Feel free to stop at the point where you have written down the characteristic system and appropriate initial conditions.
Solution: Writing the equation as

$$F(x, y, u, u_x, u_y) = (1 + x^2 + y^2)^2(u_x^2 + u_y^2) - 1 = 0,$$

and taking $k = 2$,

$$H(x, y, z, p, q, r) = r^k F(x, y, z, -p/r, -q/r) = (1+x^2+y^2)^2(p^2+q^2)-r^2.$$

The characteristic equations are, as usual,

$$x_t = H_p, \quad y_t = H_q, \quad z_t = H_r,$$

$$p_t = -H_x, \quad q_t = -H_y, \quad r_t = -H_z.$$

With this H,

$$x_t = 2(1 + x^2 + y^2)^2 p, \quad y_t = (1 + x^2 + y^2)^2 q, \quad z_t = -2r,$$

$$p_t = -4(1+x^2+y^2)(p^2+q^2)x, \quad q_t = -4(1+x^2+y^2)(p^2+q^2)y, \quad r_t = 0.$$

We need to solve these equations with some initial conditions

$$x(s,0) = x_0(s), \quad y(s,0) = y_0(s), \quad z(s,0) = z_0(s),$$

$$p(s,0) = p_0(s), \quad q(s,0) = q_0(s), \quad r(s,0) = r_0(s)$$

which should satisfy

$$y_0(s) = z_0(s) = 0.$$

In addition they should be admissible, *i.e.* they should satisfy

$$(1 + x_0(s)^2 + y_0(s)^2)^2(p_0(s)^2 + q_0(s)^2) - r_0(s)^2 = 0$$

and

$$p_0(s)\frac{dx_0(s)}{ds} + q_0(s)\frac{dy_0(s)}{ds} + r_0(s)\frac{dz_0(s)}{ds} = 0.$$

We also want $u_y = -q/r > 0$ initially. The choice of parameterisation is arbitrary, but we may as well take $x_0(s) = s$. Then

$$x_0 = s, \quad y_0 = 0, \quad z_0 = 0, \quad p_0 = 0, \quad r_0 = \pm(1 + s^2)q_0.$$

The negative sign is required to ensure $u_y > 0$ in initially. One such choice of admissible data is

$$x_0 = s, \quad y_0 = 0, \quad z_0 = 0, \quad p_0 = 0, \quad q_0 = -1, \quad r_0 = 1 + s^2.$$

I will skip the remaining calculations, since you were not asked for them, but the final answer is

$$u(x,y) = \frac{1}{2}\arcsin\frac{2y}{1 + x^2 + y^2}.$$

2. Find the Euler-Lagrange equation for the Lagrangian

$$L(x, u, u_x, u_{xx}) = uu_x - u_x u_{xx}.$$

Simplify as much as possible. What is going on here?
Solution: For a general second order Lagrangian with one dependent and one independent variable the Euler-Lagrange equation is

$$\frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial L}{\partial u_{xx}} = 0.$$

In this case

$$\frac{\partial L}{\partial u} = u_x, \quad \frac{\partial L}{\partial u_x} = u - u_{xx}, \quad \frac{\partial L}{\partial u_{xx}} = -u_x,$$

so the Euler-Lagrange equation is

$$u_x - \frac{d}{dx}(u - u_{xx}) + \frac{d^2}{dx^2}(-u_x) = 0,$$

or, after simplifying,

$$0 = 0.$$

Thus *all* functions satisfy the Euler-Lagrange equations. This may seem surprising at first, but

$$\int_a^b L(x, u, u_x, u_{xx})\, dx = \int_a^b (uu_x - u_x u_{xx})\, dx = \int_a^b \frac{d}{dx} \frac{u^2 - u_x^2}{2}\, dx$$

so, by the Fundamental Theorem of Calculus,

$$\int_a^b L(x, u, u_x, u_{xx})\, dx = \frac{u(b)^2 - u_x(b)^2}{2} - \frac{u(a)^2 - u_x(a)^2}{2}.$$

Variations of u which fix u and u_x at the endpoints therefore leave $\int_a^b L(x, u, u_x, u_{xx})\, dx$ unchanged, so *every* choice of u is stationary. This explains why all u satisfy the Euler-Lagrange equations.

3. As discussed in lecture, the minimal surface problem, with Lagrangian

$$L(x, y, z, z_x, z_y) = \sqrt{1 + z_x^2 + z_y^2}$$

has six independent symmetries, the three translations

$$(x, y, z) \mapsto (x + s, y, z),$$

$$(x, y, z) \mapsto (x, y + s, z),$$

$$(x, y, z) \mapsto (x, y, z + s),$$

and the three rotations

$$(x, y, z) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z),$$

$$(x, y, z) \mapsto (x \cos\theta - z \sin\theta, y, x \sin\theta + z \cos\theta),$$

$$(x, y, z) \mapsto (x, y \cos\theta - z \sin\theta, y \sin\theta + z \cos\theta).$$

For each of these, find the infinitesimal generator and the corresponding conserved current.

Solution: To find the infinitesimal generator we just differentiate with respect to the parameter and then set the parameter to zero. In this was we get, in order,

$$(\Delta x, \Delta y, \Delta z) = (1, 0, 0),$$

$$(\Delta x, \Delta y, \Delta z) = (0, 1, 0),$$
$$(\Delta x, \Delta y, \Delta z) = (0, 0, 1),$$
$$(\Delta x, \Delta y, \Delta z) = (-y, x, 0),$$
$$(\Delta x, \Delta y, \Delta z) = (-z, 0, x),$$
$$(\Delta x, \Delta y, \Delta z) = (0, -z, y).$$

To find the conserved currents we specialise the general formula for conserved currents for first order variational problems,

$$P_j = L\Delta x_j + \sum_{\lambda=1}^{\mu} \frac{\partial L}{\partial u_{\lambda,j}} \Delta u_\lambda - \sum_{\lambda=1}^{\mu} \sum_{k=1}^{n} u_{\lambda,k} \frac{\partial L}{\partial u_{\lambda;j}} \Delta x_k,$$

to the case of one dependent variable z and two independent variables x and y:

$$P_x = \left(L - z_x \frac{\partial L}{\partial z_x}\right) \Delta x - z_y \frac{\partial L}{\partial z_x} \Delta y + \frac{\partial L}{\partial z_x} \Delta z,$$

$$P_y = -z_x \frac{\partial L}{\partial z_y} \Delta x + \left(L - z_y \frac{\partial L}{\partial z_y}\right) \Delta y + \frac{\partial L}{\partial z_y} \Delta z.$$

For the particular Lagrangian given above,

$$\frac{\partial L}{\partial z_x} = \frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}, \quad \frac{\partial L}{\partial z_y} = \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}},$$

and all the other partial derivatives of L are zero, so we get

$$P_x = \frac{(1 + z_y^2)\Delta x - z_x z_y \Delta y + z_x \Delta z}{\sqrt{1 + z_x^2 + z_y^2}},$$

$$P_y = \frac{-z_x z_y \Delta x + (1 + z_x^2)\Delta y + z_y \Delta z}{\sqrt{1 + z_x^2 + z_y^2}}.$$

Substituting the infinitesimal generators found previously, we get the conserved currents

$$(P_x, P_y) = \frac{(1 + z_y^2, -z_x z_y)}{\sqrt{1 + z_x^2 + z_y^2}},$$

$$(P_x, P_y) = \frac{(-z_x z_y, 1 + z_x^2)}{\sqrt{1 + z_x^2 + z_y^2}},$$

$$(P_x, P_y) = \frac{(z_x, z_y)}{\sqrt{1 + z_x^2 + z_y^2}},$$

$$(P_x, P_y) = \frac{(-y(1 + z_y^2) - x z_x z_y, y z_x z_y + x(1 + z_x^2))}{\sqrt{1 + z_x^2 + z_y^2}},$$

$$(P_x, P_y) = \frac{(-z(1 + z_y^2) + x z_x, z z_x z_y + x z_y)}{\sqrt{1 + z_x^2 + z_y^2}},$$

$$(P_x, P_y) = \frac{(z z_x z_y + y z_x, -z(1 + z_x)^2 + y z_y)}{\sqrt{1 + z_x^2 + z_y^2}}.$$