MA 342H
Assignment 1
Due 12 February 2014

`Id: 342H-s2014-1.m4,v 1.1 2014/02/05 11:14:01 john Exp john`

1. Solve the initial value problem

$$xu_x + yu_y = x + y + u, \quad u(x, 1) = 0.$$

Solution: The characteristic equations are

$$dx/ds = x, \quad dy/ds = y, \quad dz/ds = x + y + z.$$

These are to be solved with initial conditons

$$x_0 = r, \quad y_0 = 1, \quad z_0 = 0.$$

The solution is

$$x = re^s, \quad y = e^s, \quad z = (1 + r)se^s$$

Eliminating r and s,

$$z = (x + y) \log y$$

so

$$u(x, y) = (x + y) \log y$$

Checking this,

$$u_x(x, y) = \log y, \quad u_y(x, y) = \log y + x/y + 1,$$

$$xu_x(x, y) + yu_y(x, y) = (x + y) \log y + x + y,$$
$$x + y + u(x + y) = x + y + (x + y) \log y,$$

so indeed

$$xu_x(x, y) + yu_y(x, y) = x + y + u(x + y).$$

The initial condition $u(x, 1) = 0$ is also clearly satisfied.

2. Solve Burgers' equation

$$u_t + uu_x = 0$$

with initial conditions

$$u(0, x) = 1/x.$$

Be careful about the choice of square root. Where is your solution valid? What happens to the solution and its partial derivatives as you approach the boundary of its domain of definition?

Solution: From the general "solution formula"

$$u = \varphi(x - ut)$$

with $\varphi(x) = 1/x$ we obtain

$$u = 1/(x - ut)$$

$$u(x - ut) = 1$$

$$tu^2 - xu + 1 = 0$$

Using the quadratic formula,

$$u(t, x) = \frac{x \pm \sqrt{x^2 - 4t}}{2t}$$

Note that the expression on the right hand side makes sense for

$$t \leq x^2/4, \quad t \neq 0.$$

The restriction $t \neq 0$ is annoying, because it prevents us from substituting $t = 0$ and comparing with the initial conditions to determine the sign in front of the square root. It is however clear that

$$\lim_{t \to 0} \sqrt{x^2 - 4t} = \sqrt{x^2} = |x|.$$

If

$$\frac{x \pm \sqrt{x^2 - 4t}}{2t}$$

is to approach a finite limit as $t \to 0$ then the numerator must tend to zero, since the denominator clearly does. The only possible choice of sign for the square root is therefore

$$u(t, x) = \begin{cases} \frac{x+\sqrt{x^2-4t}}{2t} & \text{if } x < 0, \\ \frac{x-\sqrt{x^2-4t}}{2t} & \text{if } x > 0. \end{cases}$$

For $t < 0$ there is a jump discontinuity at $x = 0$, since

$$\lim_{x \to 0^-} u(t,x) = (-t)^{-1/2}, \quad \lim_{x \to 0^+} u(t,x) = -(-t)^{-1/2}.$$

As $t \to 0^+$ the magnitude of the jump becomes infinite, which is not surprising in view of the singularity of the initial data at $x = 0$. To see what is happening at $t = 0$ it is better to write the function u defined above in the equivalent form

$$u(t,x) = \begin{cases} \frac{2}{x - \sqrt{x^2 - 4t}} & \text{if } x < 0, \\ \frac{2}{x + \sqrt{x^2 - 4t}} & \text{if } x > 0. \end{cases}$$

The expression on the right hand side is well defined and continuously differentiable for

$$t < x^2/4, \quad x \neq 0.$$

Extending u in this way we see that $u(0,x) = 1/x$, as desired. At the boundary of the domain of definition

$$\lim_{t \to (x^2/4)^-} u(t,x) = 2/x,$$

which is finite for $x \neq 2$. Also,

$$u_t(t,x) = \begin{cases} -\frac{u(t,x)^2}{\sqrt{x^2 - 4t}} & \text{if } x < 0, \\ \frac{u(t,x)^2}{\sqrt{x^2 - 4t}} & \text{if } x > 0, \end{cases}$$

$$u_x(t,x) = \begin{cases} \frac{u(t,x)}{\sqrt{x^2 - 4t}} & \text{if } x < 0, \\ -\frac{u(t,x)}{\sqrt{x^2 - 4t}} & \text{if } x > 0. \end{cases}$$

There are other ways to write these derivatives, but these expressions have the advantage that they show immediately that

$$u_t(t,x) + u(t,x) u_x(t,x) = 0$$

everywhere and

$$\lim_{t \to (x^2/4)^-} u_t(t,x) = \begin{cases} -\infty & \text{if } x < 0 \ , \\ \infty & \text{if } x > 0 \ , \end{cases}$$

$$\lim_{t \to (x^2/4)^-} u_t(t,x) = -\infty.$$