MA 342H
Assignment 3
Due 2 April

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1. The usual way to solve the heat equation

$$u_t - \kappa u_{xx} = 0$$

numerically is to approximate the t derivative with a forward difference and x derivatives with centred differences:[1]

$$\frac{U(t+k,x) - U(t,x)}{k} - \kappa\frac{U(t,x+h) - 2U(t,x) + U(t,x-h)}{h^2} = 0.$$

As with the finite difference scheme for the wave equation discussed in class, this scheme may or may not exhibit numerical instability, depending on the values of κ, h and k. Determine, using the same method used in class for the wave equation, where the stability threshold lies.

2. The differential equation

$$y_{xx} = \alpha y^2 + \beta y$$

in an interval $[a, b]$ is the Euler-Lagrange equation for the first order Lagrangian

$$L(x, y, y_x) = \frac{1}{2}y_x^2 + \frac{\alpha}{3}y^3 + \frac{\beta}{2}y^2.$$

The equation can be solved exactly in terms of elliptic functions, but one can also solve it numerically, either by finite differences or finite elements. Derive the equations for a finite element scheme with continuous piecewise linear elements.

[1] We write U in place of u to distinguish the approximate solution from the exact solution.