MA 2326
Assignment 6
Due 3 April 2014

`Id: 2326-s2014-6.m4,v 1.1 2014/03/27 16:25:33 john Exp john`

1. The equilibria of the autonomous system

$$x' = 2x + y - 2x^3 \quad y' = 2x - y - 2x^3$$

are $(-1, 0)$, $(0, 0)$ and $(1, 0)$. Show that

$$V(x, y) = x^4 - 2x^2 + y^2$$

is a strict Lyapunov function for the equilibria $(-1, 0)$ and $(1, 0)$.
Solution: Clearly V is continuously differentiable. Computing derivatives,

$$\frac{\partial V}{\partial x} = 4x^3 - 4x, \quad \frac{\partial V}{\partial y} = 2y,$$

$$\frac{\partial^2 V}{\partial x^2} = 12x^2 - 4, \quad \frac{\partial^2 V}{\partial x \partial y} = 0, \quad \frac{\partial^2 V}{\partial y^2} = 2,$$

At $(-1, 0)$ and $(1, 0)$ the first derivatives are zero and the Hessian

$$\begin{pmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} \\ \frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial y^2} \end{pmatrix}$$

is positive definite, so the points $(-1, 0)$ and $(1, 0)$ are strict local minima.

$$V' = \frac{\partial V}{\partial x} x' + \frac{\partial V}{\partial y} y' = -8x^6 + 16x^4 - 8x^2 - 2y^2 = -8x^2(x^2 - 1)^2 - 2y^2$$

This strictly less than zero except at $(-1, 0)$, $(0, 0)$ and $(1, 0)$.

2. For which of the following is the origin a stable equilibrium? For which is it strictly stable?

   (a)

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Solution: The characteristic polynomial is

$$\lambda^3 - 15\lambda^2 - 18\lambda = \lambda\left(\lambda - \frac{15}{2} - \frac{1}{2}\sqrt{297}\right)\left(\lambda - \frac{15}{2} + \frac{1}{2}\sqrt{297}\right).$$

The eigenvalues are 0, $\frac{15}{2} + \frac{1}{2}\sqrt{297}$, and $\frac{15}{2} - \frac{1}{2}\sqrt{297}$. One of these, $\frac{15}{2} + \frac{1}{2}\sqrt{297}$, has positive real part, so the origin is not a stable equilibrium and hence not a strictly stable equilibrium.

   (b)

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 3 & -2 \\ -3 & -1 & 1 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Solution: The characteristic polynomial is

$$\lambda^3 + 3\lambda^2 + 17\lambda + 15 = (\lambda + 1)(\lambda + 1 - i\sqrt{14})(\lambda + 1 + i\sqrt{14}).$$

The eigenvalues are -1, $-1 + i\sqrt{14}$, and $-1 + i\sqrt{14}$. All of these have negative real part, so the origin is a strictly stable equilibrium, and hence a stable equilibrium.

   (c)

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -9 & 12 & 0 \\ 12 & -16 & 0 \\ 20 & 15 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Solution: The characteristic polynomial is

$$\lambda^3 + 25\lambda^2 = \lambda^2(\lambda + 25).$$

The eigenvalues are 0, with multiplicity 2 and -25 with multiplicity 1. The real part of 0 is not negative, so there is no hope of strict stability. All eigenvalues have non-positive real part, so there will be strict stability provided that for each eigenvalue either the real part is negative or the algebraic and geometric multiplicities are equal. The eigenvalue -25 is fine, both because its real part is negative and because its algebraic and geometric multiplicities agree. The eigenvalue 0, however, does not satisfy either condition. Its real part is non-negative, its algebraic multiplicity is 2, since the characteristic polynomial has a factor of λ^2, and its geometric multiplicity is 1, since the zero eigenspace, which is just the nullspace, is one dimensional, with basis

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

So the origin is not stable.