MA 2326
Assignment 5
Due 20 March 2014

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1. Find the equilibria of the autonomous system

$$x'(t) = x(x^3 - 2y^3), \quad y' = y(2x^3 - y^3).$$

Solution: An equilibrium is a point (x, y) where

$$x(x^3 - 2y^3) = 0, \quad y(2x^3 - y^3) = 0.$$

The first of these expressions will be zero if either $x = 0$ or $x^3 - 2y^3$ and the second will be zero if either $y = 0$ or $2x^3 - y^3 = 0$. There are then four possibilities:

$$x = 0, \quad y = 0,$$

or

$$x = 0, \quad 2x^3 - y^3 = 0$$

or

$$x^3 - 2y^3 = 0, \quad y = 0$$

or

$$x^3 - 2y^3 = 0, \quad 2x^3 - y^3 = 0.$$

Each of these has $(x, y) = (0, 0)$ as its only solution, so that is the only equilibrium.

2. Suppose that U and V are open subsets of $\mathbf{R}^m$ and that $\varphi: U \to V$ and $\psi: V \to U$ are continuously differentiable and that ψ is the inverse of φ. Suppose also that $F: U \to \mathbf{R}^m$ and that $G: V \to \mathbf{R}^m$ is defined by

$$G_j(y) = \sum_{k=1}^{m} \frac{\partial \varphi_j}{\partial x_k}(\psi(y)) F_k(\psi(y)).$$

(a) Prove that $x\colon I \to U$ is a solution to the autonomous system

$$x'(t) = F(x(t))$$

if and only if $y\colon I \to V$, defined by

$$y(t) = \varphi(x(t)),$$

is a solution to the autonomous system

$$y'(t) = G(y(t)).$$

I is an interval.
Solution: If $y(t) = \varphi(x(t))$ then, by the chain rule,

$$y_j'(t) = \sum_{k=1}^{m} \frac{\partial \varphi_j}{\partial x_k}(x(t)) x_k'(t).$$

If x satisfies $x'(t) = F(x(t))$ then

$$y_j'(t) = \sum_{k=1}^{m} \frac{\partial \varphi_j}{\partial x_k}(x(t)) F_k(x(t)) = \left(\sum_{k=1}^{m} \frac{\partial \varphi_j}{\partial x_k} F_k \right) (\psi(y(t))),$$

or, in view of the definition of G,

$$y_j'(t) = G_j(y(t)).$$

Conversely,

$$x(t) = \psi(\varphi(x(t))) = \psi(y(t))$$

so, by the chain rule,

$$x_k'(t) = \sum_{j=1}^{m} \frac{\partial \psi_k}{\partial y_j}(y(t)) y_j'(t)$$

If y satisfies $y'(t) = G(y(t))$ then

$$x_k'(t) = \sum_{j=1}^{m} \frac{\partial \psi_k}{\partial y_j}(y(t)) G_j(y(t)) = \left(\sum_{j=1}^{m} \frac{\partial \psi_k}{\partial y_j} G_j \right) (\varphi(x(t))).$$

Differentiating $\psi(\varphi(x)) = x$ by the chain rule

$$\sum_{j=1}^{m} \frac{\partial \psi_k}{\partial y_j}(\varphi(x)) \frac{\partial \varphi_j}{\partial x_l}(x) = \begin{cases} 1 & \text{if } k = l \\ 0, & \text{if } k \neq l \ . \end{cases}$$

Applying this to $x = \psi(y)$,

$$\sum_{j=1}^{m} \frac{\partial \psi_k}{\partial y_j}(y) \frac{\partial \varphi_j}{\partial x_l}(\psi(y)) = \begin{cases} 1 & \text{if } k = l \\ 0, & \text{if } k \neq l \ . \end{cases}$$

Multiplying the definition of $G_j(y)$ by $\partial \psi_l / \partial y_j(y)$ and summing over j,

$$\sum_{j=1}^{m} G_j(y) \frac{\partial \psi_l}{\partial y_j}(y) = \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial \varphi_j}{\partial x_k}(\psi(y)) \frac{\partial \psi_l}{\partial y_j}(y) F_k(\psi(y)) = F_l(\psi(y)),$$

so

$$\left(\sum_{j=1}^{m} \frac{\partial \psi_k}{\partial y_j} G_j \right) (\varphi(x(t))) = F_k(x(t)).$$

and

$$x_k'(t) = F_k(x(t)).$$

(b) Prove that ξ is an equilibrium of $x'(t) = F(x(t))$ if and only if $\eta = \varphi(\xi)$ is an equilibrium of $y'(t) = G(y(t))$.
Solution: If ξ is an equilibrium then $F_k(\xi) = 0$ for $1 \leq k \leq m$ and

$$G_j(\eta) = \sum_{k=1}^{m} \frac{\partial \varphi_j}{\partial x_k}(\psi(\eta)) F_k(\psi(\eta)) = \sum_{k=1}^{m} \frac{\partial \varphi_j}{\partial x_k}(\xi) F_k(\xi) = 0$$

for $1 \leq j \leq m$, so η is an equilibrium. Conversely, if η is an equilibrium then $G_j(\eta) = 0$ and, using again the relation

$$F_l(\psi(y)), = \sum_{j=1}^{m} G_j(y) \frac{\partial \psi_l}{\partial y_j}(y),$$

$F(\xi) = 0$, so ξ is an equilibrium.

(c) Prove that ξ is a stable equilibrium if and only if η is.
Solution: Suppose ξ is a stable equilibrium For any $\epsilon > 0$ there is, by the continuity of φ, an $\epsilon' > 0$ such that

$$\|x - \xi\| < \epsilon'$$

implies

$$\|\varphi(x) - \eta\| < \epsilon.$$

Because ξ is a stable equilibrium, there is a $\delta' > 0$ such that

$$\|x(0) - \xi\| < \delta'$$

implies

$$\|x(t) - \xi\| < \epsilon'$$

for all $t \geq 0$. There is also, by the continuity of ψ, a $\delta > 0$ such that

$$\|y - \eta\| < \delta$$

implies

$$\|\psi(y) - \xi\| < \delta'.$$

Combining these, if

$$\|y(0) - \eta\| < \delta$$

then

$$\|x(0) - \xi\| = \|\psi(y(0)) - \xi\| < \delta',$$
$$\|x(t) - \xi\| < \epsilon',$$

and

$$\|y(t) - \eta\| = \|\varphi(x(t)) - \eta\| < \epsilon.$$

There is such a δ for every ϵ, so η is a stable equilibrium. To prove the converse, switch x and y, ξ and η, and φ and ψ in the argument above.

(d) Prove that ξ is a strictly stable equilibrium if and only if η is.
Solution: Because φ is continuous,

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \varphi(y(t)) = \varphi\left(\lim_{t\to+\infty} x(t)\right),$$

so

$$\lim_{t\to+\infty} y(t) = \eta$$

if

$$\lim_{t\to+\infty} x(t) = \xi.$$

So η is strictly stable if ξ is. To prove the converse, again switch x and y, ξ and η, and φ and ψ.