MA 2326
Assignment 4
Due 6 March 2014

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1. Given that

$$x_1(t) = t^2$$

and

$$x_2(t) = t^3$$

are linearly independent solutions of the differential equation

$$t^2 x''(t) - 4tx'(t) + 6x(t) = 0$$

in the interval $(0, \infty)$, find the fundamental matrix for

$$A(t) = \begin{pmatrix} 0 & 1 \\ -6t^{-2} & 4t^{-1} \end{pmatrix}.$$

Solution: The vector valued functions

$$\mathbf{v}_1(t) = \begin{pmatrix} x_1(t) \\ x_1'(t) \end{pmatrix}, \quad \mathbf{v}_2(t) = \begin{pmatrix} x_2(t) \\ x_2'(t) \end{pmatrix}$$

are solutions of the vector differential equation

$$\mathbf{v}'(t) = A(t)\mathbf{v}(t),$$

so

$$V(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} = \begin{pmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{pmatrix}$$

satisfies the matrix differential equation $V'(t) = A(t)V(t)$ and

$$W(s,t) = V(t)V(s)^{-1} = \begin{pmatrix} 3s^{-2}t^2 - 2s^{-3}t^3 & -s^{-1}t^2 + s^{-2}t^3 \\ 6s^{-2}t - 6s^{-3}t^2 & -2s^{-1}t + 3s{-}2t^2 \end{pmatrix}.$$

is the fundamental matrix.

2. Given that
$$x(t) = t$$
is a solution to the differential equation
$$t^2 x''(t) - t x'(t) + x(t) = 0$$
on the interval $(0, \infty)$, find a second solution by Wronskian reduction of order.
Solution: By the general theory we know that
$$w(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$$
satisfies the first order differential equation
$$t^2 w'(t) - t w(t) = 0,$$
so
$$w(t) = w(1) \exp\left(\int_1^t \frac{ds}{s}\right) = w(1)\exp(\log t) = w(1)t.$$
Substituting,
$$t x_2'(t) - x_2(t) = w(1)t,$$
or
$$x_2'(t) - t^{-1} x_2(t) = w(1),$$
so
$$\begin{aligned} x_2(t) &= x_2(1) \exp\left(\int_1^t \frac{ds}{s}\right) + \int_1^t \exp\left(\int_s^t \frac{dr}{r}\right) w(1)\, ds \\ &= x_2(1)t + \int_1^t w(1)\frac{t}{s}\, ds \\ &= x_2(1)t + w(1)t\log t. \end{aligned}$$
You can choose $w(1)$ and $x_2(1)$ arbitrarily, as long as $w(1) \neq 0$.

3. Given that
$$W(s,t) = \begin{pmatrix} \cos\log(s/t) & -s\sin\log(s/t) \\ t^{-1}\sin\log(s/t) & st^{-1}\cos\log(s/t) \end{pmatrix}$$
is the fundamental matrix for
$$A(t) = \begin{pmatrix} 0 & 1 \\ -t^{-2} & -t^{-1} \end{pmatrix}$$
in the interval $(0, \infty)$, solve the initial value problem
$$x(t_0) = x_0 \quad x'(t_0) = v_0$$

inhomogeneous linear differential equation

$$t^2 x''(t) + tx'(t) + x(t) = f(t).$$

Solution: We write the differential equation as a system

$$\begin{pmatrix} x'(t) \\ v'(t) \end{pmatrix} = A(t) \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} + b(t)$$

where A is as given above and

$$\mathbf{b}(t) = \begin{pmatrix} 0 \\ t^{-2} f(t) \end{pmatrix}.$$

The general formula

$$\mathbf{x}(t) = W(t_0, t)\mathbf{x}(t_0) + \int_{t_0}^{t} W(s, t)\mathbf{b}(s)\, ds$$

gives

$$x(t) = x_0 \cos\log(t_0/t) - v_0 \sin\log(t_0/t) - \int_{t_0}^{t} s^{-1} \sin\log(s/t) f(s)\, ds.$$