MA 2326
Assignment 3
Due 20 February 2014

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1. Solve the initial value problem

$$x' = y + z, \quad y' = x + z, \quad z' = x + y,$$

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0.$$

Solution: In matrix form, the equation is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

so the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \exp(tA) \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

A[1] Jordan decomposition of A is

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

[1] *A*, not *the*, because various choices are possible, although they all lead to the same result.

It follows that

$$\exp(tA) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

Multiplying,

$$\exp(tA) = \begin{pmatrix} \frac{e^{2t}+2e^t}{3} & \frac{e^{2t}-e^t}{3} & \frac{e^{2t}-e^t}{3} \\ \frac{e^{2t}-e^t}{3} & \frac{e^{2t}+2e^t}{3} & \frac{e^{2t}-e^t}{3} \\ \frac{e^{2t}-e^t}{3} & \frac{e^{2t}-e^t}{3} & \frac{e^{2t}+2e^t}{3} \end{pmatrix},$$

so

$$x(t) = \frac{e^{2t}+2e^t}{3}x_0 + \frac{e^{2t}-e^t}{3}y_0 + \frac{e^{2t}-e^t}{3}z_0,$$
$$y(t) = \frac{e^{2t}-e^t}{3}x_0 + \frac{e^{2t}+2e^t}{3}y_0 + \frac{e^{2t}-e^t}{3}z_0,$$
$$z(t) = \frac{e^{2t}-e^t}{3}x_0 + \frac{e^{2t}-e^t}{3}y_0 + \frac{e^{2t}+2e^t}{3}z_0.$$

A less messy way to write this is

$$x(t) = \frac{x_0+y_0+z_0}{3}e^{2t} + \frac{2x_0-y_0-z_0}{3}e^{-t},$$
$$y(t) = \frac{x_0+y_0+z_0}{3}e^{2t} + \frac{-x_0+2y_0-z_0}{3}e^{-t},$$
$$z(t) = \frac{x_0+y_0+z_0}{3}e^{2t} + \frac{-x_0-y_0+2z_0}{3}e^{-t}.$$

2. Find the solution of the initial value problem $x(0) = x_0$, $x'(0) = y_0$ for the forced harmonic oscillator in the critically damped case:

$$x''(t) + 2rx'(t) + r^2x(t) = \cos(\Omega t).$$

Solution: Applying $d^2/dt^2 + \Omega^2$ to both sides,

$$x''''(t) + 2rx'''(t) + (r^2+\Omega^2)x''(t) + 2r\Omega^2x'(t) + r^2\Omega^2x(t) = 0.$$

The characteristic polynomial is

$$\lambda^4 + 2r\lambda^3 + (r^2+\Omega^2)\lambda^2 + 2r\Omega^2\lambda + r^2\Omega^2 = (\lambda+r)^2(\lambda-i\Omega)(\lambda+i\Omega).$$

The eigenvalues are therefore $-r$, with multiplicity 2, and $i\Omega$, and $-i\Omega$, each with multiplicity 1. A basis for solutions is therefore

$$\{e^{-rt}, te^{-rt}, e^{i\Omega t}, e^{-i\Omega t}\}.$$

A more convenient basis is

$$\{e^{-rt}, te^{-rt}, \cos(\Omega t), \sin(\Omega t)\}.$$

Any solution is therefore of the form

$$x(t) = \alpha e^{-rt} + \beta te^{-rt} + \gamma\cos(\Omega t) + \delta\sin(\Omega t).$$

Differentiating,

$$x'(t) = (\beta - r\alpha)e^{-rt} - r\beta te^{-rt} + \Omega\delta\cos(\Omega t) - \Omega\gamma\sin(\Omega t)$$

and

$$x''(t) = (r^2\alpha - 2\beta r)e^{-rt} + r^2\beta te^{-rt} - \Omega^2\gamma\cos(\Omega t) - \Omega^2\delta\sin(\Omega t).$$

Then

$$x''(t) - 2rx'(t) + r^2x(t) = [(r^2 - \Omega^2)\gamma - 2r\Omega\delta]\cos(\Omega t) + [2r\Omega\gamma + (r^2 - \Omega^2)\delta]\sin(\Omega t).$$

This is equal to $\cos(\Omega t)$ if and only if

$$\gamma = \frac{(r^2 - \Omega^2)}{(r^2 + \Omega^2)^2}, \quad \delta = \frac{2r\Omega}{(r^2 + \Omega^2)^2}.$$

To find α and β we substitute $t = 0$ in the equations for $x(t)$ and $x'(t)$:

$$x_0 = \alpha + \gamma, \quad y_0 = \beta - r\alpha + \Omega\delta,$$

so

$$\alpha = x_0 - \gamma = x_0 - \frac{(r^2 - \Omega^2)}{(r^2 + \Omega^2)^2}$$

$$\beta = y_0 + r\alpha - \Omega\delta = y_0 + rx_0 - r\gamma - \Omega\delta = y_0 + rx_0 - \frac{r}{(r^2 + \Omega^2)}.$$

Substituting these values of α, β, γ and δ into

$$x(t) = \alpha e^{-rt} + \beta te^{-rt} + \gamma\cos(\Omega t) + \delta\sin(\Omega t),$$

we find

$$x(t) = x_0(e^{-rt} + rte^{-rt}) + y_0te^{-rt} + \frac{(r^2 - \Omega^2)(\cos(\Omega t) - e^{-rt}) - (r^2 + \Omega^2)rtr^{-rt} + 2r\Omega\sin(\Omega t)}{(r^2 + \Omega^2)^2}.$$