MA 2326
Assignment 1
Due 6 February 2014

`Id: 2326-s2014-1.m4,v 1.2 2014/02/07 14:36:28 john Exp john`

1. For each of the following equations or systems give the order and state whether it is linear or not. Do not attempt to solve.

   (a)

   $$dx/dt = -y, \quad dy/dt = x$$

   Solution: First order linear.

   (b)

   $$\left(\frac{dy}{dx}\right)^2 = 4y^3 - g_2 y - g_3$$

   Solution: First order nonlinear.

   (c)

   $$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0$$

   Solution: Second order linear.

   (d)

   $$\frac{d^2x}{dt^2} + (x^2-1)\frac{dx}{dt} + x = 0$$

   Solution: Second order nonlinear.

2. Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 - 4x - 1}{2y}, \quad y(0) = \sqrt{2}.$$

Solution: The equation is separable. Multiplying by $2y\,dx$,

$$2y\,dy = (3x^2 - 4x - 1)\,dx.$$

Integrating from the point $(0, \sqrt{2})$,

$$y^2 - 2 = x^3 - 2x^2 - x$$

or

$$y^2 = x^3 - 2x^2 - x + 2 = (x+1)(x-1)(x-2).$$

Since $y(0) > 0$ we should take the positive square root:

$$y(x) = \sqrt{(x+1)(x-1)(x-2)}.$$

The solution is well defined in the closed interval $[-1, 1]$, but is only differentiable in the open interval $(-1, 1)$.

3. Find *all* solutions[1] of the differential equation

$$\frac{dy}{dx} = y^2 - 1.$$

Solution: This equation is also separable. Multiplying by dx and dividing by $y^2 - 1$,

$$\frac{dy}{y^2 - 1} = dx.$$

Note that

$$\frac{dy}{y^2 - 1} = \frac{1}{2}\frac{dy}{y-1} - \frac{1}{2}\frac{dy}{y+1}.$$

Integrating from (x_0, y_0),

$$\frac{1}{2}\log\left(\frac{y-1}{y+1}\right) - \frac{1}{2}\log\left(\frac{y_0-1}{y_0+1}\right) = x - x_0,$$

or

$$\frac{1}{2}\log\left(\frac{y-1}{y+1}\right) - x = \frac{1}{2}\log\left(\frac{y_0-1}{y_0+1}\right) - x_0.$$

[1] Do not list the same solution more than once.

Multipling by two and exponentiating,

$$\frac{y-1}{y+1}e^{-2x} = \frac{y_0-1}{y_0+1}e^{-2x_0}.$$

The right hand side is constant. Call it c. Then

$$\frac{y-1}{y+1} = ce^{2x}$$

and

$$y = \frac{1+ce^{2x}}{1-ce^{2x}}.$$

These are solutions, but they are not the only solutions. The formal calculation above made some assumptions about y, and lost solutions which fail to satisfy those assumptions. In dividing by y^2-1 we ignore the possibilities

$$y = 1$$

and

$$y = -1,$$

both of which are perfectly good solutions. Purely by luck we didn't lose $y=1$, which corresponds to $c=0$ in the family of solutions found previously, but we need to add $y=-1$ to the list. Even worse, the choice of antidervative for

$$\frac{1}{2}\frac{dy}{y-1} - \frac{1}{2}\frac{dy}{y+1}$$

implicitly assumed

$$\frac{y-1}{y+1} > 0$$

If

$$\frac{y-1}{y+1} < 0$$

then we should instead have found

$$\frac{1}{2}\log\left(\frac{1-y}{y+1}\right) - \frac{1}{2}\log\left(\frac{1-y_0}{y_0+1}\right) = x - x_0,$$

or

$$\frac{1}{2}\log\left(\frac{1-y}{y+1}\right) - x = \frac{1}{2}\log\left(\frac{1-y_0}{y_0+1}\right) - x_0,$$

$$\frac{1-y}{y+1}e^{-2x} = \frac{1-y_0}{y_0+1}e^{-2x_0}.$$

The right hand side is again constant. Call it C. Then

$$\frac{1-y}{y+1} = Ce^{2x}$$

and

$$y = \frac{1-Ce^{2x}}{1+Ce^{2x}}.$$

As it happens, we already found these solutions. They are the previous family of solutions with $c = -C$, so we didn't in fact lose any solutions in this way. The complete set of solutions is then

$$y = -1, \quad y = \frac{1+ce^{2x}}{1-ce^{2x}}.$$