MA 3426
Assignment 4
Due never

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1. Solve the initial value problem

$$u(0,x) = \sqrt{1+x^2}$$

for Burgers' Equation for $-1 < t < 1$. What is

$$\lim_{t\to 1^-} u(t,x)?$$

Solution:

$$u = f(x-ut)$$

where

$$f(x)^2 = 1 + x^2,$$

so

$$u^2 = 1 + (x-ut)^2$$

or

$$(1-t^2)u^2 + 2txu - 1 - x^2 = 0.$$

From the quadratic formula,

$$u = \frac{\sqrt{1-t^2+x^2} - tx}{1-t^2}.$$

The choice of the positive sign for the square root is determined by the requirement that $u(0,x) > 0$.

$$\lim_{t\to 1^-} \sqrt{1-t^2+x^2} = \sqrt{x},$$

so
$$\lim_{t\to 1^-}(\sqrt{1-t^2+x^2}-tx)=\begin{cases}-2x & \text{if } x\le 0\\ 0 & \text{if } x\ge 0.\end{cases}$$
Since the denominator $1-t^2$ tends to zero,
$$\lim_{t\to 1^-} u(t,x)=\infty$$
if $x<0$. If $x>0$ we can either use L'Hôpital's rule or write it as
$$u=\frac{1+x^2}{\sqrt{1-t^2+x^2}+tx},$$
from which it follows that
$$\lim_{t\to 1^-} u(t,x)=\frac{1+x^2}{2x}$$
for $x>0$.

2. Prove that
$$u(t,x)=\begin{cases}-1 & \text{if } -1<x<1-t,\\ (x-1)/t & \text{if } \max(1-t,1+t-\sqrt{8t})<x<1+t,\\ 1 & \text{otherwise}\end{cases}$$
is an *admissible* solution of Burgers' Equation for $t>0$.
Solution:

There are three things to check:

- Where u is continuously differentiable, $u_t+uu_x=0$.
- Along any curves where u is not continuously differentiable, the jump condition is satisfied: Either $u_{\text{left}}=u_{\text{right}}$ or
$$\frac{dx}{dt}=\frac{u_{\text{left}}+u_{\text{right}}}{2}.$$
- Along any curves where u is not continuously differentiable, the entropy condition is satisfied: $u_{\text{left}}\ge u_{\text{right}}$.

The first is completely trivial where u is constant, and is straighforward in the region where $u=(x-1)/t$. The curves along which we need to check jump and entropy conditions are
$$\begin{aligned}C_1&=\{(t,x)\in\mathbf{R}^2\colon 0<t<2, x=-1\}\\ C_2&=\{(t,x)\in\mathbf{R}^2\colon 0<t<2, x=1-t\}\\ C_3&=\{(t,x)\in\mathbf{R}^2\colon t>0, x=1+t\}\\ C_4&=\{(t,x)\in\mathbf{R}^2\colon t>0, x=1+t-\sqrt{8t}\}\end{aligned}$$

On C_2 and C_3, $u_{\text{left}} = u_{\text{right}}$, so both the jump and entropy conditions are satisfied. On C_1, $u_{\text{left}} = 1$, u_{right} and $dx/dt = 0$, so again both the jump and entropy conditions are satisfied. The most complicated is C_4, where

$$u_{\text{left}} = 1, \quad u_{\text{right}} = \frac{x-1}{t} = \frac{t-\sqrt{8t}}{t} = 1 - \sqrt{\frac{8}{t}}$$

and

$$\frac{dx}{dt} = \frac{d}{dt}\left(1 + t - \sqrt{8t}\right) = 1 - \sqrt{\frac{2}{t}}.$$

Both the jump condition

$$\frac{dx}{dt} = \frac{u_{\text{left}} + u_{\text{right}}}{2}.$$

and the entropy condition $u_{\text{left}} \geq u_{\text{right}}$ are satisfied.