MA 3426
Assignment 3
Due 19 March 2013

`Id: 3426-s2013-3.m4,v 1.1 2013/03/12 15:12:36 john Exp john`

1. Show that

$$u(x,y) = \frac{1}{4\pi}\log(x^2+y^2)$$

is a fundamental solution to the Laplace Equation in $\mathbf{R}^2$.
Hint: Write $\langle u_{xx}+u_{yy},\varphi\rangle$ as an integral. Split this integral into three pieces: one where $x^2+y^2<r_1^2$, one where $r_1^2 \le x^2+y^2 \le r_2^2$, and one where $r_2^2 < x^2+y^2$. Apply Green's Second Identity to the second of these, and let r_1 and r_2 tend to 0 and ∞, respectively.
Solution:

First, note that u, though not integrable, is locally integrable and hence defines a distribution. Indeed,

$$\int_{x^2+y^2\le R^2} |u| = \int_0^R |\log r|\, r\, dr = \begin{cases} \frac{1}{4}R^2 - \frac{1}{2}R^2\log R & \text{if } R \le 1, \\ \frac{1}{2}R^2\log R - \frac{1}{4}R^2 + \frac{1}{2} & \text{if } R \ge 1. \end{cases}$$

By definition,

$$\langle u_{xx}+u_{yy},\varphi\rangle = \langle u, \varphi_{xx}+\varphi_{yy}\rangle = \int_{(x,y)\in\mathbf{R}^2} u(x,y)\operatorname{div}\operatorname{grad}\varphi(x,y)$$

for $\varphi \in \mathcal{D}(\mathbf{R}^2)$. As suggested, we split this into three integrals, I_1, I_2, and I_3, over the sets $x^2+y^2<r_1^2$, $r_1^2 \le x^2+y^2 \le r_2^2$, and $r_2^2 < x^2+y^2$. Now

$$\begin{aligned} I_1 &= \int_{x^2+y^2<r_1^2} u(x,y)\operatorname{div}\operatorname{grad}\varphi(x,y) \\ &\le \max_{(x,y)\in\mathbf{R}^2} |\operatorname{div}\operatorname{grad}\varphi(x,y)| \int_{x^2+y^2<r_1^2} |u(x,y)|, \end{aligned}$$

which tends to 0 as r_1 as r_1 tends to 0^+. By the definition of $\mathcal{D}(\mathbf{R}^2)$, there is a ρ such that

$$\varphi(x,y)=0$$

whenever $x^2+y^2\geq\rho^2$. Differentiating,

$$\operatorname{div}\operatorname{grad}\varphi(x,y)=0$$

whenever $x^2+y^2\geq\rho^2$, and hence

$$I_3=\int_{r_2^2<x^2+y^2}u(x,y)\operatorname{div}\operatorname{grad}\varphi(x,y)=0$$

if $r_2>\rho$. It follows then that I_3 tends to 0 as r_2 tends to ∞.

Thus, in the limit, only I_2 survives. For this integral we use, as suggested, Greens' Second Identity.

$$\begin{aligned}I_2&=\int_{r_1^2\leq x^2+y^2\leq r_2^2}u(x,y)\operatorname{div}\operatorname{grad}\varphi(x,y)\\&=\int_{x^2+y^2=r_2^2}\left(u\frac{\partial\varphi}{\partial r}-\varphi\frac{\partial u}{\partial r}\right)-\int_{x^2+y^2=r_1^2}\left(u\frac{\partial\varphi}{\partial r}-\varphi\frac{\partial u}{\partial r}\right)\\&\quad-\int_{r_1^2\leq x^2+y^2\leq r_2^2}\varphi(x,y)\operatorname{div}\operatorname{grad}u(x,y).\end{aligned}$$

The first integral on the right is zero once $r_2>\rho$ and the last integral on the right is zero because u is harmonic for $r_1^2\leq x^2+y^2\leq r_2^2$, so I_2 simplifies to

$$I_2=\int_{x^2+y^2=r_1^2}\left(\varphi\frac{\partial u}{\partial r}-u\frac{\partial\varphi}{\partial r}\right)=\frac{1}{2\pi r_1}\int_{x^2+y^2=r_1^2}\varphi-\frac{\log r_1}{2\pi}\int_{x^2+y^2=r_1^2}\frac{\partial\varphi}{\partial r}.$$

The first integral on the right is the average value of φ over the circle of radius r_1, which tends to $\varphi(0,0)$ as r_1 tends to 0^+. The second integral tends to 0, because

$$\left|\int_{x^2+y^2=r_1^2}\frac{\partial\varphi}{\partial r}\right|\leq 2\pi r_1\max_{(x,y)\in\mathbf{R}^2}\|\operatorname{grad}\varphi(x,y)\|$$

and $\lim_{r_1\to0^+}r_1\log r_1=0$. Thus

$$\langle u_{xx}+u_{yy},\varphi\rangle=I_1+I_2+I_3=\varphi(0,0)=\langle\delta,\varphi\rangle\,.$$

Since this is true for all $\varphi\in\mathcal{D}(\mathbf{R}^2)$,

$$u_{xx}+u_{yy}=\delta$$

as distributions and u is a fundamental solution.

2. Solve the initial value problem $u(t,x) = f(x)$ for the first order linear scalar equation
$$(t^2+1)u_t + (1+x^2)u_x = 0.$$
For which (t,x) is this a classical solution?
Solution: The characteristic equations are:
$$dt/ds = 1+t^2, \quad dx/ds = 1+x^2$$
with solution
$$\arctan t - \arctan t_0 = s = \arctan x - \arctan x_0,$$
from which it follows that
$$\arctan x - \arctan t = \arctan x_0 - \arctan t_0$$
and
$$\frac{x-t}{1+xt} = \frac{x_0-t_0}{1-x_0t_0}.$$
Choosing the initial point (t_0,x_0) such that $t_0 = 0$,
$$x_0 = \frac{x-t}{1+xt}$$
and
$$u(t,x) = u(0,x_0) = f(x_0) = f\left(\frac{x-t}{1+xt}\right).$$
This solution is valid for $1+xt > 0$.

3. Show that each of the following is a symmetry of Burgers' equation:

   (a) $\overline{u} = u$, $\overline{t} = t+\tau$, $\overline{x} = x+\xi$,
   Solution:
$$u_t = \overline{u}_t = \overline{u}_{\overline{t}}\overline{t}_t + \overline{u}_{\overline{x}}\overline{x}_t = \overline{u}_{\overline{t}},$$
$$u = \overline{u},$$
   and
$$u_x = \overline{u}_x = \overline{u}_{\overline{t}}\overline{t}_x + \overline{u}_{\overline{x}}\overline{x}_x = \overline{u}_{\overline{x}},$$
   so
$$\overline{u}_{\overline{t}} + \overline{u}\overline{u}_{\overline{x}} = u_t + uu_x.$$

(b) $\overline{u} = -u$, $\overline{t} = -t$, $\overline{x} = x$,
Solution:

$$u_t = -\overline{u}_t = -\overline{u}_{\overline{t}}\overline{t}_t - \overline{u}_{\overline{x}}\overline{x}_t = \overline{u}_{\overline{t}},$$

$$u = -\overline{u},$$

and

$$u_x = -\overline{u}_x = -\overline{u}_{\overline{t}}\overline{t}_x - \overline{u}_{\overline{x}}\overline{x}_x = -\overline{u}_{\overline{x}},$$

so

$$\overline{u}_{\overline{t}} + \overline{u}\overline{u}_{\overline{x}} = u_t + uu_x.$$

(c) $\overline{u} = 1/u$, $\overline{t} = x$, $\overline{x} = t$,
Solution:

$$u_t = (1/\overline{u})_t = (-1/\overline{u}^2)(\overline{u}_{\overline{t}}\overline{t}_t + \overline{u}_{\overline{x}}\overline{x}_t) = (-1/\overline{u}^2)\overline{u}_{\overline{x}},$$

$$u = 1/\overline{u},$$

and

$$u_x = (1/\overline{u})_x = (-1/\overline{u}^2)(\overline{u}_{\overline{t}}\overline{t}_x + \overline{u}_{\overline{x}}\overline{x}_x) = (-1/\overline{u}^2)\overline{u}_{\overline{t}},$$

so

$$\overline{u}_{\overline{t}} + \overline{u}\overline{u}_{\overline{x}} = (-1/\overline{u}^3)(u_t + uu_x).$$

(d) $\overline{u} = u + v$, $\overline{t} = t$, $\overline{x} = x + vt$.
Solution:

$$u_t = \overline{u}_t = \overline{u}_{\overline{t}}\overline{t}_t + \overline{u}_{\overline{x}}\overline{x}_t = \overline{u}_{\overline{t}} + vu_{\overline{x}},$$

$$u = \overline{u} - v,$$

and

$$u_x = \overline{u}_x = \overline{u}_{\overline{t}}\overline{t}_x + \overline{u}_{\overline{x}}\overline{x}_x = \overline{u}_{\overline{x}},$$

so

$$\overline{u}_{\overline{t}} + \overline{u}\overline{u}_{\overline{x}} = u_t + uu_x.$$