MA 3426
Assignment 3
Due 19 March 2013

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Id: 3426-s2013-3.m4,v 1.1 2013/03/12 15:12:36 john Exp john
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1. Show that
$$u(x,y) = \frac{1}{4\pi} \log(x^2 + y^2)$$
is a fundamental solution to the Laplace Equation in $\mathbf{R}^2$.
Hint: Write $\langle u_{xx} + u_{yy}, \varphi \rangle$ as an integral. Split this integral into three pieces: one where $x^2 + y^2 < r_1^2$, one where $r_1^2 \le x^2 + y^2 \le r_2^2$, and one where $r_2^2 < x^2 + y^2$. Apply Green's Second Identity to the second of these, and let r_1 and r_2 tend to 0 and ∞, respectively.
Solution:

First, note that u, though not integrable, is locally integrable and hence defines a distribution. Indeed,
$$\int_{x^2+y^2 \le R^2} |u| = \int_0^R |\log r| \, r \, dr = \begin{cases} \frac{1}{4}R^2 - \frac{1}{2}R^2 \log R & \text{if } R \le 1, \\ \frac{1}{2}R^2 \log R - \frac{1}{4}R^2 + \frac{1}{2} & \text{if } R \ge 1. \end{cases}$$
By definition,
$$\langle u_{xx} + u_{yy}, \varphi \rangle = \langle u, \varphi_{xx} + \varphi_{yy} \rangle = \int_{(x,y) \in \mathbf{R}^2} u(x,y) \operatorname{div} \operatorname{grad} \varphi(x,y)$$
for $\varphi \in \mathcal{D}(\mathbf{R}^2)$. As suggested, we split this into three integrals, I_1, I_2, and I_3, over the sets $x^2 + y^2 < r_1^2$, $r_1^2 \le x^2 + y^2 \le r_2^2$, and $r_2^2 < x^2 + y^2$. Now
$$\begin{aligned} I_1 &= \int_{x^2+y^2<r_1^2} u(x,y) \operatorname{div} \operatorname{grad} \varphi(x,y) \\ &\le \max_{(x,y) \in \mathbf{R}^2} |\operatorname{div} \operatorname{grad} \varphi(x,y)| \int_{x^2+y^2<r_1^2} |u(x,y)|, \end{aligned}$$

which tends to 0 as r_1 as r_1 tends to 0^+. By the definition of $\mathcal{D}(\mathbf{R}^2)$, there is a ρ such that

$$\varphi(x, y) = 0$$

whenever $x^2 + y^2 \geq \rho^2$. Differentiating,

$$\operatorname{div}\operatorname{grad}\varphi(x, y) = 0$$

whenever $x^2 + y^2 \geq \rho^2$, and hence

$$I_3 = \int_{r_2^2 < x^2+y^2} u(x, y) \operatorname{div}\operatorname{grad}\varphi(x, y) = 0$$

if $r_2 > \rho$. It follows then that I_3 tends to 0 as r_2 tends to ∞.

Thus, in the limit, only I_2 survives. For this integral we use, as suggested, Greens' Second Identity.

$$\begin{aligned} I_2 &= \int_{r_1^2 \leq x^2+y^2 \leq r_2^2} u(x, y) \operatorname{div}\operatorname{grad}\varphi(x, y) \\ &= \int_{x^2+y^2=r_2^2} \left(u\frac{\partial \varphi}{\partial r} - \varphi\frac{\partial u}{\partial r} \right) - \int_{x^2+y^2=r_1^2} \left(u\frac{\partial \varphi}{\partial r} - \varphi\frac{\partial u}{\partial r} \right) \\ &\quad - \int_{r_1^2 \leq x^2+y^2 \leq r_2^2} \varphi(x, y) \operatorname{div}\operatorname{grad} u(x, y). \end{aligned}$$

The first integral on the right is zero once $r_2 > \rho$ and the last integral on the right is zero because u is harmonic for $r_1^2 \leq x^2 + y^2 \leq r_2^2$, so I_2 simplifies to

$$I_2 = \int_{x^2+y^2=r_1^2} \left(\varphi\frac{\partial u}{\partial r} - u\frac{\partial \varphi}{\partial r} \right) = \frac{1}{2\pi r_1} \int_{x^2+y^2=r_1^2} \varphi - \frac{\log r_1}{2\pi} \int_{x^2+y^2=r_1^2} \frac{\partial \varphi}{\partial r}.$$

The first integral on the right is the average value of φ over the circle of radius r_1, which tends to $\varphi(0, 0)$ as r_1 tends to 0^+. The second integral tends to 0, because

$$\left| \int_{x^2+y^2=r_1^2} \frac{\partial \varphi}{\partial r} \right| \leq 2\pi r_1 \max_{(x,y)\in\mathbf{R}^2} \|\operatorname{grad}\varphi(x, y)\|$$

and $\lim_{r_1 \to 0^+} r_1 \log r_1 = 0$. Thus

$$\langle u_{xx} + u_{yy}, \varphi \rangle = I_1 + I_2 + I_3 = \varphi(0, 0) = \langle \delta, \varphi \rangle .$$

Since this is true for all $\varphi \in \mathcal{D}(\mathbf{R}^2)$,

$$u_{xx} + u_{yy} = \delta$$

as distributions and u is a fundamental solution.

2. Solve the initial value problem $u(t,x) = f(x)$ for the first order linear scalar equation
$$(t^2+1)u_t + (1+x^2)u_x = 0.$$
For which (t,x) is this a classical solution?
Solution: The characteristic equations are:
$$dt/ds = 1+t^2, \quad dx/ds = 1+x^2$$
with solution
$$\arctan t - \arctan t_0 = s = \arctan x - \arctan x_0,$$
from which it follows that
$$\arctan x - \arctan t = \arctan x_0 - \arctan t_0$$
and
$$\frac{x-t}{1+xt} = \frac{x_0-t_0}{1-x_0t_0}.$$
Choosing the initial point (t_0, x_0) such that $t_0 = 0$,
$$x_0 = \frac{x-t}{1+xt}$$
and
$$u(t,x) = u(0,x_0) = f(x_0) = f\left(\frac{x-t}{1+xt}\right).$$
This solution is valid for $1+xt > 0$.

3. Show that each of the following is a symmetry of Burgers' equation:

   (a) $\overline{u} = u$, $\overline{t} = t+\tau$, $\overline{x} = x+\xi$,
   Solution:
$$u_t = \overline{u}_t = \overline{u}_{\overline{t}}\overline{t}_t + \overline{u}_{\overline{x}}\overline{x}_t = \overline{u}_{\overline{t}},$$
$$u = \overline{u},$$
   and
$$u_x = \overline{u}_x = \overline{u}_{\overline{t}}\overline{t}_x + \overline{u}_{\overline{x}}\overline{x}_x = \overline{u}_{\overline{x}},$$
   so
$$\overline{u}_{\overline{t}} + \overline{u}\overline{u}_{\overline{x}} = u_t + uu_x.$$

(b) $\overline{u} = -u$, $\overline{t} = -t$, $\overline{x} = x$,
Solution:

$$u_t = -\overline{u}_t = -\overline{u}_{\overline{t}}\overline{t}_t - \overline{u}_{\overline{x}}\overline{x}_t = \overline{u}_{\overline{t}},$$

$$u = -\overline{u},$$

and

$$u_x = -\overline{u}_x = -\overline{u}_{\overline{t}}\overline{t}_x - \overline{u}_{\overline{x}}\overline{x}_x = -\overline{u}_{\overline{x}},$$

so

$$\overline{u}_{\overline{t}} + \overline{u}\overline{u}_{\overline{x}} = u_t + uu_x.$$

(c) $\overline{u} = 1/u$, $\overline{t} = x$, $\overline{x} = t$,
Solution:

$$u_t = (1/\overline{u})_t = (-1/\overline{u}^2)(\overline{u}_{\overline{t}}\overline{t}_t + \overline{u}_{\overline{x}}\overline{x}_t) = (-1/\overline{u}^2)\overline{u}_{\overline{x}},$$

$$u = 1/\overline{u},$$

and

$$u_x = (1/\overline{u})_x = (-1/\overline{u}^2)(\overline{u}_{\overline{t}}\overline{t}_x + \overline{u}_{\overline{x}}\overline{x}_x) = (-1/\overline{u}^2)\overline{u}_{\overline{t}},$$

so

$$\overline{u}_{\overline{t}} + \overline{u}\overline{u}_{\overline{x}} = (-1/\overline{u}^3)(u_t + uu_x).$$

(d) $\overline{u} = u + v$, $\overline{t} = t$, $\overline{x} = x + vt$.
Solution:

$$u_t = \overline{u}_t = \overline{u}_{\overline{t}}\overline{t}_t + \overline{u}_{\overline{x}}\overline{x}_t = \overline{u}_{\overline{t}} + vu_{\overline{x}},$$

$$u = \overline{u} - v,$$

and

$$u_x = \overline{u}_x = \overline{u}_{\overline{t}}\overline{t}_x + \overline{u}_{\overline{x}}\overline{x}_x = \overline{u}_{\overline{x}},$$

so

$$\overline{u}_{\overline{t}} + \overline{u}\overline{u}_{\overline{x}} = u_t + uu_x.$$