MA 3426
Assignment 2
Due 5 March 2013

`Id: 3426-s2013-2.m4,v 1.3 2013/03/12 15:14:50 john Exp john`

1. (a) Write the explicit solution to the initial value problem

$$u_{tt} - c^2 u_{xx} - c^2 u_{yy} = 0$$
$$u(0, x, y) = f(x, y) \quad u_t(0, x, y) = g(x, y)$$

for the Wave Equation in $\mathbf{R}^{1+2}$ in terms of convolutions,

$$u(t, \cdot) = \frac{\partial}{\partial t}(S(t, \cdot) \star f) + S(t, \cdot) \star g$$

with an explicit function $S(t, x, y)$.
Hint: Start from the Cartesian form

$$\begin{aligned} u(t, x, y) = {} & \frac{\partial}{\partial t}\left[\frac{1}{4\pi c}\int_{B_{ct,(x,y)}} (c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-1/2} f(\xi, \eta)\, d\xi\, d\eta\right] \\ & + \frac{1}{4\pi c}\int_{B_{ct,(x,y)}} (c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-1/2} g(\xi, \eta)\, d\xi\, d\eta \end{aligned}$$

of the explicit solution, not the polar form. Here $B_{ct,(x,y)}$ is the ball of radius ct about (x, y), i.e.,

$$B_{ct,(x,y)} = \{(\xi, \eta) \in \mathbf{R}^2 \colon (x-\xi)^2 + (y-\eta)^2 < c^2t^2\}.$$

Solution: The solution formula above is

$$u(t, \cdot) = \frac{\partial}{\partial t}(S(t, \cdot) \star f) + S(t, \cdot) \star g$$

with

$$S(t, x, y) = \begin{cases} A(c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-1/2} & \text{if } x^2 + y^2 < c^2t^2 \\ 0 & \text{otherwise,} \end{cases}$$

$$A = (4\pi c)^{-1/2}.$$

(b) Show that, if $f = 0$,

$$\|u(t,\cdot)\|_r \le C t^\alpha \|g\|_q$$

if $\frac{1}{q} - \frac{1}{r} \le \frac{1}{2}$, with constants C and α which depend only on q and r. You should find α, at least, explicitly.
Hint: Use Young's inequality. If you don't know what a Beta integral is you should probably find out.
Solution: By Young's inequality,

$$\|u(t,\cdot)\|_r \le \|S(t,\cdot)\|_p \|g\|_q.$$

Now

$$\|S(t,\cdot)\|_p^p = \int_{(x,y)\subset B_{ct}} A^p (c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-p/2}\, dx\, dy$$

or, in polar coordinates,

$$\|S(t,\cdot)\|_p^p = \int_{-\pi}^{\pi} \int_0^{ct} (c^2t^2 - r^2)^{-p/2}\, r\, dr\, d\theta = 2\pi \int_0^{ct} (c^2t^2 - r^2)^{-p/2}\, r\, dr.$$

Making the change of variable

$$s = \sqrt{\frac{r}{ct}},$$

$$\|S(t,\cdot)\|_p^p = \pi (ct)^{2-p/2} \int_0^1 (1-s)^{-p/2} ds$$

In terms of Beta and Gamma,

$$\|S(t,\cdot)\|_p^p = \pi (ct)^{2-p/2} B(1-p/2, 1) = \pi (ct)^{2-p/2} \frac{\Gamma(1-p/2)\Gamma(1)}{\Gamma(2-p/2)}$$

or

$$\|S(t,\cdot)\|_p^p = \pi (ct)^{2-p/2} (1-p/2)^{-1}.$$

Thus

$$\|u(t,\cdot)\|_r \le C t^\alpha \|g\|_q$$

with

$$\alpha = \frac{2}{p} - \frac{1}{2}$$

and

$$C = A \pi^{1/p} c^\alpha (1-p/2)^{-1/p}.$$

2. Show that if p is a polynomial then so is $p \star \varphi$ for any φ in $\mathcal{D}(\mathbf{R}^n)$, where p is considered as a distribution in $\mathcal{D}'(\mathbf{R}^n)$.
Solution: If p is a polynomial of degree n then
$$\partial^\alpha p = 0$$
for any multiindex α with $|\alpha| > n$. It was shown in lecture that
$$\partial^\alpha (p \star \varphi) = (\partial^\alpha p) \star \varphi,$$
so
$$\partial^\alpha (p \star \varphi) = 0.$$
Any function with this property is, by Taylor's Theorem, a polynomial of degree at most n.

3. (a) Show that $w\colon \mathcal{D}(\mathbf{R}) \to \mathbf{R}$, defined by
$$\langle w, \varphi \rangle = \int_{x\in\mathbf{R}} e^{-|x|}\varphi(x),$$
is a distribution.
Solution:
$w(x) = e^{-|x|}$ is integrable, and integration against an integrable, or even just locally integrable, function always gives a distribution.

(b) Compute, as distributions,

i. dw/dx,
Solution: By definition,
$$\begin{aligned}\langle dw/dx, \varphi \rangle &= -\langle w, d\varphi/dx \rangle \\ &= -\int_{x\in\mathbf{R}} e^{-|x|}\varphi'(x) \\ &= -\int_0^\infty e^{-x}\varphi'(x)\,dx - \int_{-\infty}^0 e^x\varphi'(x)\,dx.\end{aligned}$$
Integrating by parts,
$$\langle dw/dx, \varphi \rangle = -\int_0^\infty e^{-x}\varphi(x)\,dx + \int_{-\infty}^0 e^x\varphi(x)\,dx = \langle v, \varphi \rangle,$$
where
$$v(x) = \begin{cases} -e^{-x} & \text{if } x > 0, \\ e^x & \text{if } x < 0. \end{cases}$$
The boundary terms in the integration by parts cancel. It follows that $dw/dx = v$ not just as functions on $\mathbf{R} - \{0\}$ but as distributions on $\mathbf{R}$.

ii. d^2w/dx,
Solution:
Similarly,

$$\left\langle d^2w/dx^2, \varphi \right\rangle = - \langle dw/dx, d\varphi/dx \rangle$$
$$= - \langle v, d\varphi/dx \rangle$$
$$= - \int_{x \in \mathbf{R}} v(x)\varphi'(x).$$
$$= \int_0^\infty e^{-x}\varphi'(x)\,dx - \int_{-\infty}^0 e^x\varphi'(x)\,dx.$$

This, when we integrate by parts, the boundary terms do not cancel,

$$\left\langle d^2w/dx^2, \varphi \right\rangle = \int_0^\infty e^{-x}\varphi(x)\,dx + \int_{-\infty}^0 e^x\varphi(x)\,dx - 2\varphi(0),$$

or

$$\left\langle d^2w/dx^2, \varphi \right\rangle = \langle w, \varphi \rangle - 2 \langle \delta, \varphi \rangle ,$$

so

$$d^2w/dx^2 = w - 2\delta.$$

iii. $d^2w/dx - w$.
Solution:
From the preceding it follows immediately that

$$d^2w/dx^2 - w = -2\delta.$$