MA 3426
Assignment 2
Due 5 March 2013

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1. (a) Write the explicit solution to the initial value problem

$$u_{tt} - c^2 u_{xx} - c^2 u_{yy} = 0$$
$$u(0,x,y) = f(x,y) \quad u_t(0,x,y) = g(x,y)$$

for the Wave Equation in $\mathbf{R}^{1+2}$ in terms of convolutions,

$$u(t,\cdot) = \frac{\partial}{\partial t}(S(t,\cdot) \star f) + S(t,\cdot) \star g$$

with an explicit function $S(t,x,y)$.
Hint: Start from the Cartesian form

$$\begin{aligned} u(t,x,y) = {} & \frac{\partial}{\partial t}\left[\frac{1}{4\pi c}\int_{B_{ct,(x,y)}} (c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-1/2} f(\xi,\eta)\, d\xi\, d\eta\right] \\ & + \frac{1}{4\pi c}\int_{B_{ct,(x,y)}} (c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-1/2} g(\xi,\eta)\, d\xi\, d\eta \end{aligned}$$

of the explicit solution, not the polar form. Here $B_{ct,(x,y)}$ is the ball of radius ct about (x,y), i.e.,

$$B_{ct,(x,y)} = \{(\xi,\eta) \in \mathbf{R}^2 \colon (x-\xi)^2 + (y-\eta)^2 < c^2t^2\}.$$

Solution: The solution formula above is

$$u(t,\cdot) = \frac{\partial}{\partial t}(S(t,\cdot) \star f) + S(t,\cdot) \star g$$

with

$$S(t,x,y) = \begin{cases} A(c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-1/2} & \text{if } x^2+y^2 < c^2t^2 \\ 0 & \text{otherwise,} \end{cases}$$

$$A = (4\pi c)^{-1/2}.$$

(b) Show that, if $f = 0$,

$$\|u(t, \cdot)\|_r \le C t^\alpha \|g\|_q$$

if $\frac{1}{q} - \frac{1}{r} \le \frac{1}{2}$, with constants C and α which depend only on q and r. You should find α, at least, explicitly.
Hint: Use Young's inequality. If you don't know what a Beta integral is you should probably find out.
Solution: By Young's inequality,

$$\|u(t, \cdot)\|_r \le \|S(t, \cdot)\|_p \|g\|_q.$$

Now

$$\|S(t, \cdot)\|_p^p = \int_{(x,y)\subset B_{ct}} A^p (c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-p/2}\, dx\, dy$$

or, in polar coordinates,

$$\|S(t, \cdot)\|_p^p = \int_{-\pi}^{\pi} \int_0^{ct} (c^2t^2 - r^2)^{-p/2}\, r\, dr\, d\theta = 2\pi \int_0^{ct} (c^2t^2 - r^2)^{-p/2}\, r\, dr.$$

Making the change of variable

$$s = \sqrt{\frac{r}{ct}},$$

$$\|S(t, \cdot)\|_p^p = \pi (ct)^{2-p/2} \int_0^1 (1-s)^{-p/2} ds$$

In terms of Beta and Gamma,

$$\|S(t, \cdot)\|_p^p = \pi (ct)^{2-p/2} B(1-p/2, 1) = \pi (ct)^{2-p/2} \frac{\Gamma(1-p/2)\Gamma(1)}{\Gamma(2-p/2)}$$

or

$$\|S(t, \cdot)\|_p^p = \pi (ct)^{2-p/2} (1-p/2)^{-1}.$$

Thus

$$\|u(t, \cdot)\|_r \le C t^\alpha \|g\|_q$$

with

$$\alpha = \frac{2}{p} - \frac{1}{2}$$

and

$$C = A \pi^{1/p} c^\alpha (1-p/2)^{-1/p}.$$

2. Show that if p is a polynomial then so is $p \star \varphi$ for any φ in $\mathcal{D}(\mathbf{R}^n)$, where p is considered as a distribution in $\mathcal{D}'(\mathbf{R}^n)$.
   Solution: If p is a polynomial of degree n then
   $$\partial^\alpha p = 0$$
   for any multiindex α with $|\alpha| > n$. It was shown in lecture that
   $$\partial^\alpha (p \star \varphi) = (\partial^\alpha p) \star \varphi,$$
   so
   $$\partial^\alpha (p \star \varphi) = 0.$$
   Any function with this property is, by Taylor's Theorem, a polynomial of degree at most n.

3. (a) Show that $w\colon \mathcal{D}(\mathbf{R}) \to \mathbf{R}$, defined by
   $$\langle w, \varphi \rangle = \int_{x\in\mathbf{R}} e^{-|x|}\varphi(x),$$
   is a distribution.
   Solution:
   $w(x) = e^{-|x|}$ is integrable, and integration against an integrable, or even just locally integrable, function always gives a distribution.

   (b) Compute, as distributions,

   i. dw/dx,
      Solution: By definition,
      $$\begin{aligned}\langle dw/dx, \varphi \rangle &= - \langle w, d\varphi/dx \rangle \\ &= - \int_{x\in\mathbf{R}} e^{-|x|}\varphi'(x) \\ &= - \int_0^\infty e^{-x}\varphi'(x)\,dx - \int_{-\infty}^0 e^{x}\varphi'(x)\,dx.\end{aligned}$$
      Integrating by parts,
      $$\langle dw/dx, \varphi \rangle = - \int_0^\infty e^{-x}\varphi(x)\,dx + \int_{-\infty}^0 e^{x}\varphi(x)\,dx = \langle v, \varphi \rangle,$$
      where
      $$v(x) = \begin{cases} -e^{-x} & \text{if } x > 0, \\ e^x & \text{if } x < 0. \end{cases}$$
      The boundary terms in the integration by parts cancel. It follows that $dw/dx = v$ not just as functions on $\mathbf{R} - \{0\}$ but as distributions on $\mathbf{R}$.

ii. d^2w/dx,
Solution:
Similarly,

$$\begin{aligned}\left\langle d^2w/dx^2, \varphi \right\rangle &= -\left\langle dw/dx, d\varphi/dx \right\rangle \\ &= -\left\langle v, d\varphi/dx \right\rangle \\ &= -\int_{x\in\mathbf{R}} v(x)\varphi'(x). \\ &= \int_0^\infty e^{-x}\varphi'(x)\,dx - \int_{-\infty}^0 e^x\varphi'(x)\,dx.\end{aligned}$$

This, when we integrate by parts, the boundary terms do not cancel,

$$\left\langle d^2w/dx^2, \varphi \right\rangle = \int_0^\infty e^{-x}\varphi(x)\,dx + \int_{-\infty}^0 e^x\varphi(x)\,dx - 2\varphi(0),$$

or

$$\left\langle d^2w/dx^2, \varphi \right\rangle = \left\langle w, \varphi \right\rangle - 2\left\langle \delta, \varphi \right\rangle,$$

so

$$d^2w/dx^2 = w - 2\delta.$$

iii. $d^2w/dx - w$.
Solution:
From the preceding it follows immediately that

$$d^2w/dx^2 - w = -2\delta.$$