MA 3426
Assignment 1
Due 5 February 2013

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1. Suppose Ω is a bounded open set in $\mathbf{R}^n$, $n > 2$ and that $\partial\Omega$ is continuously differentiable. A *Green's* function for Ω is a function $G: \overline{\Omega} \times \overline{\Omega} - \Delta \to \mathbf{R}$, where $\Delta = \{(\mathbf{x}, \mathbf{y}) \in \overline{\Omega} \times \overline{\Omega}: \mathbf{x} = \mathbf{y}\}$, satisfying the following conditions:

   - If $\mathbf{x} \in \partial\Omega$ then $G(\mathbf{x}, \mathbf{y}) = 0$.
   - For each fixed $\mathbf{y} \in \Omega$ there is a harmonic function v on $\overline{\Omega}$ such that
   $$G(\mathbf{x}, \mathbf{y}) = v(\mathbf{x}) + w(\mathbf{x})$$
   for $\mathbf{x} \neq \mathbf{y}$ where $w(\mathbf{x}) = c_n \|\mathbf{x} - \mathbf{y}\|^{2-n}$, $c_n = (2-d)^{-1}\omega_{n-1}^{-1}$ and ω_{n-1} is the $n-1$ dimensional measure of the unit sphere in $\mathbf{R}^n$.

   (a) Prove that such a set Ω has at most one Green's function.
   Note: This is not hard.
   Solution: Suppose G_1 and G_2 are Green's function, with corresponding harmonic functions v_1 and v_2 as above. Then $v = v_1 - v_2$ is harmonic in $\overline{\Omega}$. On $\partial\Omega$,
   $$v = v_1 - v_2 = v_1 + w - v_2 - w = G_1 - G_2 = 0.$$
   By the uniqueness of solutions to the Dirichlet problem, $v = 0$ throughout Ω. Thus $v_1 = v_2$ and so $G_1 = G_2$.

   (b) Find a Green's function for the unit ball $\{\mathbf{x} \in \mathbf{R}^n: \|\mathbf{x}\| < 1\}$.
   Note: We essentially did this in lecture.
   Solution:

$$G(\mathbf{x}, \mathbf{y}) = c_n \|\mathbf{x} - y\|^{2-n} - c_n \|\mathbf{x}\|^{2-n} \left\| \frac{\mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} - \mathbf{y} \right\|^{2-n}$$

satisfies the conditions above, with

$$v(\mathbf{x}) = -c_n \|\mathbf{x}\|^{2-n} \left\| \frac{\mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} - \mathbf{y} \right\|^{2-n}.$$

2. Supposing that Ω has a Green's function, find an integral representation for the value of a harmonic function u at an interior point $y \in \Omega$ in terms of its values on the boundary $\partial\Omega$.
Note: This was done, in the special case of the unit ball, in lecture. The argument given there works in general.
Solution: We apply Green's Second Identity twice, once with with the Ω and v above,

$$\int_{\Omega} (u \operatorname{div} \operatorname{grad} v - v \operatorname{div} \operatorname{grad} u) = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right),$$

and once with Ω and v replaced by $\Omega - \overline{B}_{\mathbf{y},\epsilon}$ and w,

$$\int_{\Omega - \overline{B}_{\mathbf{y},\epsilon}} (u \operatorname{div} \operatorname{grad} w - w \operatorname{div} \operatorname{grad} u) = \int_{\partial(\Omega - \overline{B}_{\mathbf{y},\epsilon})} \left(u \frac{\partial w}{\partial n} - w \frac{\partial u}{\partial n} \right).$$

Since u and v are harmonic in $\overline{\Omega}$ and w is harmonic in $\Omega - \overline{B}_{\mathbf{y},\epsilon}$, the left hand sides of both equations are zero.

$$\int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) = 0$$

and

$$\int_{\partial(\Omega - \overline{B}_{\mathbf{y},\epsilon})} \left(u \frac{\partial w}{\partial n} - w \frac{\partial u}{\partial n} \right) = 0.$$

The boundary of $\Omega - \overline{B}_{\mathbf{y},\epsilon}$, with the usual orientation, the one which appears in Green's Second Identity, is the boundary of Ω, also with the usual orientation, together with the sphere $\Sigma_{\mathbf{y},\epsilon}$, with the opposite orientiation. The previous equation can therefore be written as

$$\int_{\partial\Omega} \left(u \frac{\partial w}{\partial n} - w \frac{\partial u}{\partial n} \right) - \int_{\Sigma_{\mathbf{y},\epsilon}} \left(u \frac{\partial w}{\partial n} - w \frac{\partial u}{\partial n} \right) = 0.$$

Adding this to the equation for u and v gives

$$\int_{\partial\Omega}\left(u\frac{\partial G}{\partial n}-G\frac{\partial u}{\partial n}\right)-\int_{\Sigma_{\mathbf{y},\epsilon}}\left(u\frac{\partial w}{\partial n}-w\frac{\partial u}{\partial n}\right)=0.$$

By hypothesis $G=0$ on $\partial\Omega$. Also w is constant on $\Sigma_{\mathbf{y},\epsilon}$ and also, because $\Sigma_{\mathbf{y},\epsilon}=\partial B_{\mathbf{y},\epsilon}$, $\int_{\Sigma_{\mathbf{y},\epsilon}}\partial u/\partial n=0$. Therefore

$$\int_{\Sigma_{\mathbf{y},\epsilon}} u\frac{\partial w}{\partial n}=\int_{\partial\Omega} u\frac{\partial G}{\partial n}.$$

Now

$$\frac{\partial w}{\partial n}=\omega_{n-1}\epsilon^{1-n}=\mu(\Sigma_{\mathbf{y},\epsilon})$$

on $\Sigma_{\mathbf{y},\epsilon}$. The left hand side of the equation above is therefore the average over $\Sigma_{\mathbf{y},\epsilon}$ of u, which, by the Mean Value Property, is $u(\mathbf{y})$. In other words,

$$u(\mathbf{y})=\int_{\partial\Omega} u\frac{\partial G}{\partial n}.$$

A little more explicitly,

$$u(\mathbf{y})=\int_{\mathbf{x}\in\partial\Omega} u(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y})\,d\mathbf{x}.$$

The normal derivative is taken with respect to the $\mathbf{x}$ variables.

3. Prove the following relations for homogeneous polynomials p in $\mathbf{R}^n$:

   (a)

$$\mathbf{x}\cdot\operatorname{grad}p(\mathbf{x})=\deg(p)p(\mathbf{x}).$$

Solution:
Any homogeneous polynomial is of the form

$$p(x_1,\ldots,x_n)=\sum_{j_1+\cdots+j_n=\deg(p)} c_{j_1,\ldots,j_n}x_1^{j_1}\cdots x_n^{j_n}.$$

Differentiating,

$$x_kp_{x_k}(x_1,\ldots,x_n)=\sum_{j_1+\cdots+j_n=\deg(p)} j_kc_{j_1,\ldots,j_n}x_1^{j_1}\cdots x_n^{j_n}.$$

Summing over k gives the equation required.

(b)

$$\operatorname{div}\operatorname{grad}(\mathbf{x}\cdot\mathbf{x}p(\mathbf{x})) - \mathbf{x}\cdot\mathbf{x}\operatorname{div}\operatorname{grad}p(\mathbf{x}) = (2n + 4\deg(p))p(\mathbf{x}).$$

Solution:

$$\operatorname{div}\operatorname{grad}(\mathbf{x}\cdot\mathbf{x}p(\mathbf{x})) = \operatorname{div}[\operatorname{grad}(\mathbf{x}\cdot\mathbf{x})p(\mathbf{x}) + (\mathbf{x}\cdot\mathbf{x})\operatorname{grad}p(\mathbf{x})]$$

and hence

$$\operatorname{div}\operatorname{grad}(\mathbf{x}\cdot\mathbf{x}p(\mathbf{x})) = \operatorname{div}\operatorname{grad}(\mathbf{x}\cdot\mathbf{x})p(\mathbf{x})) + 2\operatorname{grad}(\mathbf{x}\cdot\mathbf{x})\cdot\operatorname{grad}p(\mathbf{x}) + (\mathbf{x}\cdot\mathbf{x})\operatorname{div}\operatorname{grad}p(\mathbf{x}).$$

Now

$$\operatorname{grad}(\mathbf{x}\cdot\mathbf{x}) = 2\mathbf{x}$$

and

$$\operatorname{div}\operatorname{grad}(\mathbf{x}\cdot\mathbf{x}) = 2n,$$

so

$$\operatorname{div}\operatorname{grad}(\mathbf{x}\cdot\mathbf{x}p(\mathbf{x})) = 2np(\mathbf{x})) + 4\mathbf{x}\cdot\operatorname{grad}p(\mathbf{x}) + (\mathbf{x}\cdot\mathbf{x})\operatorname{div}\operatorname{grad}p(\mathbf{x}).$$

The equation from the previous part then gives what we want.

4. With some clever algebra and the equations from the previous problem, it is possible to show that every homogenuous polynomial p of degree d in $\mathbf{R}^n$ can be written in the form

$$p(\mathbf{x}) = q(\mathbf{x}) + (\mathbf{x}\cdot\mathbf{x})r(\mathbf{x})$$

for a unique *harmonic* polynomial q of degree d and a unique polynomial r of degree $d - 2$. Assume this is so.

(a) What is the dimension of the vector space of all homogeneous polynomials of degree d in n variables?
Solution:
The quickest way to compute this is to consider the power series

$$\prod_{j=1}^{n}(1 - tx_k)^{-1} = \prod_{j=1}^{n}\sum_{k=0}^{\infty} t^k x_j^k$$

in $n+1$ variables. Expanding the product of sums, each monomial of degree d appears once and only once, multiplied by a factor t^d. If we substitute $x_1 = x_2 = \cdots = x_n = 1$ we get a summand of t^d for each such monomial, and these monomials form a basis for the space $P_{d,n}$ of homogeneous polynomials of degree d in the n variables $x_1, \ldots, x_n$. Thus

$$(1-t)^{-n} = \sum_{d=0}^{\infty} \dim(P_{d,n}) t^d.$$

By the extended binomial theorem,

$$\dim(P_{d,n}) = \frac{(n+d-1)!}{(n-1)!d!}$$

(b) What is the dimension of the vector space of all homogeneous harmonic polynomials of degree d in n variables?
Solution:
Let $H_{d,n}$ be the space of harmonic polynomials of degree d in $x_1, \ldots, x_n$. We know that $T\colon H_{d,n} \oplus P_{d-2,n} \to P_{d,n}$, defined by

$$(T(q,r))(\mathbf{x}) = q(\mathbf{x}) + (\mathbf{x} \cdot \mathbf{x}) r(\mathbf{x}),$$

is an isomorphism, so

$$\dim(H_{d,n}) + \dim(P_{d-2,n}) = \dim(P_{d,n})$$

and, using the result from the previous part,

$$\dim(H_{d,n}) = \frac{(n+d-1)!}{(n-1)!d!} - \frac{(n+d-3)!}{(n-3)!d!} = \frac{(n+d-3)!}{(n-1)!(d-1)!}(2n+d-3).$$