MA 3426
Assignment 2
Due 5 March 2013

`Id: 3426-s2013-2.m4,v 1.3 2013/03/12 15:14:50 john Exp john`

1. (a) Write the explicit solution to the initial value problem

$$u_{tt} - c^2 u_{xx} - c^2 u_{yy} = 0$$
$$u(0, x, y) = f(x, y) \quad u_t(0, x, y) = g(x, y)$$

for the Wave Equation in $\mathbf{R}^{1+2}$ in terms of convolutions,

$$u(t, \cdot) = \frac{\partial}{\partial t}(S(t, \cdot) \star f) + S(t, \cdot) \star g$$

with an explicit function $S(t, x, y)$.
Hint: Start from the Cartesian form

$$u(t, x, y) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c}\int_{B_{ct,(x,y)}} (c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-1/2} f(\xi, \eta)\, d\xi\, d\eta\right] + \frac{1}{4\pi c}\int_{B_{ct,(x,y)}} (c^2t^2 - (x-\xi)^2 - (y-\eta)^2)^{-1/2} g(\xi, \eta)\, d\xi\, d\eta$$

of the explicit solution, not the polar form. Here $B_{ct,(x,y)}$ is the ball of radius ct about (x, y), i.e.,

$$B_{ct,(x,y)} = \{(\xi, \eta) \in \mathbf{R}^2 \colon (x-\xi)^2 + (y-\eta)^2 < c^2t^2\}.$$

(b) Show that, if $f = 0$,

$$\|u(t, \cdot)\|_r \leq Ct^\alpha \|g\|_q$$

if $\frac{1}{q} - \frac{1}{r} \leq \frac{1}{2}$, with constants C and α which depend only on q and r. You should find α, at least, explicitly.
Hint: Use Young's inequality. If you don't know what a Beta integral is you should probably find out.

2. Show that if p is a polynomial then so is $p \star \varphi$ for any φ in $\mathcal{D}(\mathbf{R}^n)$, where p is considered as a distribution in $\mathcal{D}'(\mathbf{R}^n)$.

3. (a) Show that $w \colon \mathcal{D}(\mathbf{R}) \to \mathbf{R}$, defined by

$$\langle w, \varphi \rangle = \int_{x \in \mathbf{R}} e^{-|x|} \varphi(x),$$

is a distribution.

(b) Compute, as distributions,

i. dw/dx,

ii. $d^2 w/dx$,

iii. $d^2 w/dx - w$.