MA 3426
Assignment 1
Due 5 February 2013

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1. Suppose Ω is a bounded open set in $\mathbf{R}^n$, $n > 2$ and that $\partial\Omega$ is continuously differentiable. A *Green's* function for Ω is a function $G\colon \overline{\Omega} \times \overline{\Omega} - \Delta \to \mathbf{R}$, where $\Delta = \{(\mathbf{x}, \mathbf{y}) \in \overline{\Omega} \times \overline{\Omega} \colon \mathbf{x} = \mathbf{y}\}$, satisfying the following conditions:

   - If $\mathbf{x} \in \partial\Omega$ then $G(\mathbf{x}, \mathbf{y}) = 0$.
   - For each fixed $\mathbf{y} \in \Omega$ there is a harmonic function v on $\overline{\Omega}$ such that
   $$G(\mathbf{x}, \mathbf{y}) = v(\mathbf{x}) + w(\mathbf{x})$$
   for $\mathbf{x} \neq \mathbf{y}$ where $w(\mathbf{x}) = c_n \|\mathbf{x} - \mathbf{y}\|^{2-n}$, $c_n = (2-d)^{-1}\omega_{n-1}^{-1}$ and ω_{n-1} is the $n-1$ dimensional measure of the unit sphere in $\mathbf{R}^n$.

   (a) Prove that such a set Ω has at most one Green's function.
   Note: This is not hard.

   (b) Find a Green's function for the unit ball $\{\mathbf{x} \in \mathbf{R}^n \colon \|\mathbf{x}\| < 1\}$.
   Note: We essentially did this in lecture.

2. Supposing that Ω has a Green's function, find an integral representation for the value of a harmonic function u at an interior point $y \in \Omega$ in terms of its values on the boundary $\partial\Omega$.
   Note: This was done, in the special case of the unit ball, in lecture. The argument given there works in general.

3. Prove the following relations for homogeneous polynomials p in $\mathbf{R}^n$:

   (a)
   $$\mathbf{x} \cdot \operatorname{grad} p(\mathbf{x}) = \deg(p) p(\mathbf{x}).$$

   (b)
   $$\operatorname{div} \operatorname{grad}(\mathbf{x} \cdot \mathbf{x} p(\mathbf{x})) - \mathbf{x} \cdot \mathbf{x} \operatorname{div} \operatorname{grad} p(\mathbf{x}) = (2n + 4 \deg(p)) p(\mathbf{x}).$$

4. With some clever algebra and the equations from the previous problem, it is possible to show that every homogenuous polynomial p of degree d in $\mathbf{R}^n$ can be written in the form

   $$p(\mathbf{x}) = q(\mathbf{x}) + (\mathbf{x} \cdot \mathbf{x}) r(\mathbf{x})$$

   for a unique *harmonic* polynomial q of degree d and a unique polynomial r of degree $d - 2$. Assume this is so.

   (a) What is the dimension of the vector space of all homogeneous polynomials of degree d in n variables?

   (b) What is the dimension of the vector space of all homogeneous harmonic polynomials of degree d in n variables?