1. State the following, including any necessary hypotheses. No proofs are required.

   (a) Hölder's Inequality

   (b) The Cauchy-Schwarz Inequality

   (c) Bessel's Identity

   (d) The Reverse Triangle Inequality

2. A Banach space is a *normed space* which is *complete*.

   (a) Define the italicised terms above.

   (b) Give an example of a Banach space, other than a finite dimenional one.

   (c) Give an example of a normed space which is *not* a Banach space.

   (d) Prove the *Banach Fixed Point Theorem*. This says that if V is a Banach space, $q < 1$ and $f \colon A \to A$ is a function[1] satisfying $\|f(x) - f(y)\| \leq q\|x - y\|$ for all $x, y \in V$ then there is a $z \in V$ such that $f(z) = z$.
   Hint: The usual proof proceeds by taking any $x_1 \in V$ and defining a sequence recursively by $x_{n+1} = f(x_n)$, and then showing that this sequence converges, and that it's limit satisfies $f(z) = z$.

3. (a) Suppose that E is a Hilbert space and L^m tends to zero in $B(H, H)$, Banach space of continuous linear transformation E from to E. Show that

   $$\lim_{m\to\infty} L^m\xi = 0$$

   for all $\xi \in E$.

   (b) Syppose that E is a Hilbert space, $L \colon E \to E$ a continuous linear transformation such that the limit $\lim_{m\to\infty} L^m$ exists in $B(H, H)$. Show that if

   $$\lim_{m\to\infty} L^m\xi = 0$$

   for all $\xi \in E$ then

   $$\lim_{m\to\infty} L^m = 0.$$

   (c) Give an example of a Hilbert space E and a continuous linear transformation $L \colon E \to E$ such that

   $$\lim_{m\to\infty} L^m\xi = 0$$

   for all $\xi \in E$, but L^m does not exist.
   Hint: An example appeared on one of your assignments.

[1] not necessarily linear

4. Let L and R be the left and right unilateral shift operators on ℓ^2,

$$(L\xi)_n = \xi_{n+1} \qquad (R\xi)_n = \begin{cases} 0 & \text{if } n = 1, \\ \xi_{n-1} & \text{if } n > 1. \end{cases}$$

(a) Is L compact?

(b) Is L symmetric?

(c) Is R symmetric?

(d) What are the eigenvalues and eigenvectors of L?

(e) What are the eigenvalues and eigenvectors of R?