# MA 3421
# Assignment 5
# Due 21 November 2012

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1. It was shown in lecture that if $\sum_{k,l} |\alpha_{k,l}|^2 < \infty$ then the linear transformation $A\colon \ell^2 \to \ell^2$ defined by $(A\xi)_k = \sum_l \alpha_{k,l}\xi_l$ is compact. The condition $\sum_{k,l} |\alpha_{k,l}|^2 < \infty$ is sufficient for compactness, but not necessary. To show this, consider the operator $A$ with

$$\alpha_{k,l} = \begin{cases} k^{-s} & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases}$$

where $s \geq 0$.

(a) For which values of $s$ is $A$ bounded?
*Solution:* For all $s \geq 0$, $\|A\| = 1$.

(b) For which values of $s$ is $A$ compact?
*Solution:* For all $s > 0$, $A$ is compact. The easiest way to see this is to exhibit $A$ as a limit of compact operators. Define operators $A_n$ by

$$(A_n\xi)_k = \begin{cases} k^{-s}\xi_k & \text{if } k < n \\ 0 & \text{if } k \geq n. \end{cases}$$

and note that

$$\|A_n - A\| = n^{-s},$$

so $\lim_{n\to\infty} A_n = A$. Each $A_n$ has a finite dimensional image, and hence is compact.

(c) For which values of $s$ is $A$ symmetric?
*Solution:* For all $s \geq 0$, $A$ is symmetric, because

$$(A\xi|\eta) = \sum_{k=1}^{\infty} k^{-s}\xi_k\overline{\eta}_k = (\xi|A\eta).$$

(d) For which values of $s$ is $\sum_{k,l} |\alpha_{k,l}|^2 < \infty$?
*Solution:* For $s > 1/2$,

$$\sum_{k,l} |\alpha_{k,l}|^2 = \sum_{k=1}^{\infty} k^{-2s} < \infty.$$

For $s \leq 1/2$ the series does not converge.

2. Let $L$ and $R$ be the left and right unilateral shift operators on $\ell^2$,

$$(L\xi)_n = \xi_{n+1} \qquad (R\xi)_n = \begin{cases} 0 & \text{if } n = 1, \\ \xi_{n-1} & \text{if } n > 1. \end{cases}$$

(a) Is $L$ compact?
*Solution:* No. Let $\xi^{(m)}$ be the sequence whose $m$'th entry is 1 and all other entries are 0. $\xi^{(m)}$ is a bounded sequence in $\ell^2$. $L\xi^{(m)} = \xi^{(m-1)}$, except for $m = 1$. This has no convergent subsequence.

(b) Is $R$ compact?
*Solution:* No. With $\xi^{(m)}$ as above $R\xi^{(m)} = \xi^{(m+1)}$ and again there is no convergent subsequence.

(c) Is $L$ symmetric?
*Solution:* No. Almost any pair of elements of $\ell^2$ provides a counterexample, *e.g.*

$$(L\xi^{(1)}|\xi^{(2)}) = 0 \neq 1 = (\xi^{(1)}|L\xi^{(2)}).$$

(d) Is $R$ symmetric?
*Solution:* No. For example,

$$(R\xi^{(1)}|\xi^{(2)}) = 1 \neq 0 = (\xi^{(1)}|R\xi^{(2)}).$$

(e) What are the eigenvalues and eigenvectors of $L$?
*Solution:* $\xi \in \ell^2$ is an eigenvector with eigenvalue $\lambda$ if $L\xi = \lambda\xi$. This happens if

$$\xi_{n+1} = \lambda\xi_n$$

for all $n$, *i.e.* if

$$\xi_n = \lambda^{n-1}\xi_1.$$

This is an eigenvector with eigenvalue $\lambda$ for any $\lambda$ with $|\lambda| < 1$ and any $\xi_1 \neq 0$. The condition $|\lambda| < 1$ is required to ensure that $\xi \in \ell^2$.

(f) What are the eigenvalues and eigenvectors of $R$?
*Solution:* $\xi \in \ell^2$ is an eigenvector with eigenvalue $\lambda$ if $R\xi = \lambda\xi$. This happens if

$$\lambda\xi_n = \begin{cases} 0 & \text{if } n = 1, \\ \xi_{n-1} & \text{if } n > 1. \end{cases}$$

By induction then $\xi_n = 0$ for all $n$, unless $\lambda = 0$. There are then *no* non-zero eigenvalues. If $\lambda = 0$ then the eigenvectors are the non-zero multiples of $\xi^{(1)}$.