MA 3421
Assignment 4
Due 14 November 2012

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1. A *sesquilear form* on an inner product space E is, by definition, a function $s\colon E \times E \to \mathbf{K}$ satisfying

$$s(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 s(x_1, y) + \alpha_2 s(x_2, y)$$

and

$$s(x, \beta_1 y_1 + \beta_2 y_2) = \overline{\beta}_1 (x, y_1) + \overline{\beta}_2 (x, y_2).$$

Here $\mathbf{K}$ is $\mathbf{R}$ or $\mathbf{C}$, depending on whether E is a real or complex inner product space. In the former case the complex conjugation is, of course, irrelevant. A sesquilinear form s is called *continuous* if $x_n \to x$ and $y_n \to y$ imply $s(x_n, y_n) \to s(x, y)$. A sesquilinear form s is called bounded when there is a $\lambda \geq 0$ such that for all x and y one has $|s(x, y)| \leq \lambda \|x\| \|y\|$. Prove that a sesquilinear form is continuous if and only if it is bounded.
Solution: Suppose s is bounded, $x_n \to x, y_n \to y$ and $\epsilon > 0$. Then

$$s(x_n, y_n) - s(x, y) = s(x_n, y_n - y) + s(x_n - x, y)$$

and hence

$$|s(x_n, y_n) - s(x, y)| \leq \lambda(\|x_n\| \|y_n - y\| + \|x_n - x\| \|y\|).$$

Convergent sequences are bounded, so $\|x_n\| \leq C$ for some C. Let $\epsilon_x = \frac{\epsilon}{2\|y\|}$ and $\epsilon_y = \frac{\epsilon}{2C}$. By the definition of convergence there are N_x and N_y such that if $n \geq N_x$ then $\|x_n - x\| < \epsilon_x$ and if $n \geq N_y$ then

$\|y_n - y\| < \epsilon_y$. Let N be the maximum of N_x and N_y. If $n \geq N$ then, combining all these inequalities,

$$|s(x_n, y_n) - s(x, y)| < \epsilon.$$

There is such an N for all $\epsilon > 0$ so $s(x_n, y_n) \to s(x, y)$.

Suppose that s is not bounded. Then for each integer n there are x_n and y_n such that

$$|s(x_n, y_n)| > n\|x_n\|\|y_n\|$$

Let

$$\xi_n = \frac{x_n}{n^{1/2}\|x_n\|}, \eta_n = \frac{y_n}{n^{1/2}\|y_n\|}$$

Then $\|\xi_n\| = \|\eta_n\| = n^{-1/2}$ so $\xi_n \to 0$ and $\eta_n \to 0$. But

$$|s(\xi_n, \eta_n)| > 1$$

so $s(\xi_n, \eta_n)$ cannot converge to $s(0, 0) = 0$. Thus s is not continuous.

2. Note that if A is a linear transformation from E to E and s is defined by

$$s(x, y) = (Ax|y)$$

then s is a sequilinear form on E. Prove that s is continuous if and only if A is.
Solution:

For linear transformation and, by the previous exercise, for sesquilinear forms continuity is equivalent to boundedness. It therefore suffices to show that s is bounded if and only if A is.

Suppose A is bounded. Then

$$|s(x, y)| = |(Ax|y)| \leq \|Ax\|\|y\| \leq \|A\|\|x\|\|y\|,$$

so $|s(x, y)| \leq \lambda\|x\|\|y\|$ with $\lambda = \|A\|$.

Conversely, suppose that s is bounded, *i.e.* that there is a $\lambda \geq 0$ such that

$$|s(x, y)| \leq \lambda\|x\|\|y\|.$$

This holds in particular for $y = Ax$,

$$|s(x, Ax)| \leq \lambda\|x\|\|Ax\|.$$

But

$$\|Ax\|^2 = (Ax|Ax) = s(x, Ax),$$

so
$$\|Ax\|^2 \leq \lambda \|x\| \|Ax\|.$$
and hence
$$\|Ax\| \leq \lambda \|x\|$$
for all x. Thus A is bounded with bound at most λ.

3. Suppose E is a Hilbert space and s is a continuous sesquilinear form. Prove that there is one and only one continuous linear transformation $A\colon E \to E$ such that
$$s(x, y) = (Ax|y).$$

Solution: For each $x \in E$ define a function $f_x\colon E \to \mathbf{K}$ by
$$f_x(y) = \overline{s(x, y)}.$$

The sesquilinearity of s implies the linearity of f_x and the boundedness of s implies the boundedness of f_x. It then follows from the Riesz Representation Theorem that there is one and only one z_x in E such that
$$f_x(y) = (y, z_x).$$
Equivalently,
$$(z_x|y) = s(x, y).$$
Let $A\colon E \to E$ be the function which takes x to z_x for each $x \in E$,
$$s(x, y) = (Ax|y).$$

The sesquilinearity of s implies the linearity of A and the boundedness of s implies that of A. Thus the existence of A is established.

Suppose now that A' is also a continuous linear functions from E to E such that
$$s(x, y) = (A'x|y).$$
Then
$$((A - A')x|y) = 0$$
for all x and y, and in particular for $y = (A - A')x$. It follows that $\|(A - A')x\| = 0$ and hence $(A - A')x = 0$. So $Ax = A'x$ for all $x \in E$. But A and A' are then the same function, *i.e.* $A' = A$. This establishes the uniqueness of A.