MA 3421
Assignment 3
Due 31 October 2012

`Id: 3421-f2012-3.m4,v 1.1 2012/10/24 15:48:01 john Exp john`

1. Show that the following normed spaces are not inner product spaces.[1]

   (a) $\ell^\infty(n)$ where $n > 1$.

   (b) $L^\infty([a, b])$.

   (c) $\mathcal{L}(H, H)$, where H is a Hilbert space of dimension at least 2.

   Hint: In each case the simplest way to proceed is to show that the parallelogram identity fails, *i.e.* to find vectors u and v such that

   $$\|u + v\|^2 + \|u - v\|^2 \neq 2(\|u\|^2 + \|v\|^2).$$

   Solution:

   (a) Let $e_{j\,j=1,\ldots,n}$ be the usual basis for $\ell^\infty(n)$. Take $u = e_1$ and $v = e_2$. Then

   $$\|u\| = \|v\| = \|u + v\| = \|u - v\| = 1.$$

   (b) Let $u(t) = (t - a)/(b - a)$ and $v(t) = (b - t)/(b - a)$. Then

   $$\|u\| = \|v\| = \|u + v\| = \|u - v\| = 1.$$

   (c) Take a linearly independent pair $\{x_1, x_2\}$ of elements of H. Gramm-Schmidt them into an orthonormal pair $\{y_1, y_2\}$. Define $u\colon H \to H$ and $v\colon H \to H$ by $u(x) = (x|y_1)y_1$ and $v(x) = (x|y_2)y_2$. Then

   $$\|u\| = \|v\| = \|u + v\| = \|u - v\| = 1.$$

[1] I don't mean that they can't be equipped with inner products. I mean that there is no inner product *for which* $\|x\| = \sqrt{(x|x)}$.

2. Find an infinite orthonormal system in $L^2([-1,1])$ consisting of *polynomials.*
Hint: Start with the obvious linearly independent set, $\{1, t, t^2, t^2, \ldots\}$ and apply the Gramm-Schmidt procedure. It's conceptually simple, if computational painful, to get the first few. The main problem is to guess the general form. If you can't do that then at least compute up to degree four.
Solution:

The first few, obtained by Gramm-Schmidt, are

$$u_0(t) = \sqrt{\frac{1}{2}}1 \quad u_1(t) = \sqrt{\frac{3}{2}}t \quad u_2(t) = \sqrt{\frac{5}{8}}(3t^2 - 1)$$

$$u_3(t) = \sqrt{\frac{7}{8}}(5t^3 - 3t) \quad u_4(t) = \sqrt{\frac{9}{32}}(35t^4 - 30t^2 + 3)$$

In general

$$u_n(t) = \sqrt{\frac{2n+1}{2}}P_n(t),$$

where P_n is the n'th Legendre polynomial.