MA 3421
Assignment 2
Due 17 October 2012

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1. This problem and all the others on this problem set relate to a pair of linear transformations, $L\colon \ell^p \to \ell^p$ and $R\colon \ell^p \to \ell^p$, called the left and right unilateral shift, respectively.[1]

$$(L\xi)_n = \xi_{n+1} \qquad (R\xi)_n = \begin{cases} 0 & \text{if } n = 1, \\ \xi_{n-1} & \text{if } n > 1. \end{cases}$$

Prove that L and R are bounded, and hence continuous, for all $1 \leq p \leq \infty$. Compute their norms.

Solution:

$$\|L\xi\|^p = \sum_{n=1}^{\infty} |(L\xi)_n|^p = \sum_{n=1}^{\infty} |\xi_{n+1}|^p = \sum_{n=2}^{\infty} |\xi_n|^p \leq \sum_{n=1}^{\infty} |\xi_{n+1}|^p = \|\xi\|^p$$

if $p < \infty$ and

$$\|L\xi\| = \sup_{n\geq 1} |(L\xi)_n| = \sup_{n\geq 1} |\xi_{n+1}| = \sup_{n\geq 2} |\xi_n| \leq \sup_{n\geq 1} |\xi_n| = \|\xi\|$$

if $p = \infty$. In either case $\|L\xi\| \leq \|\xi\|$, so L is bounded. To show that $\|L\| = 1$ we need to know that there is no smaller value of λ such that $\|L\xi\| \leq \lambda\|\xi\|$ for all ξ. This is clear if, for example, we take

$$\xi_n = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{if } n \neq 2, \end{cases}$$

or, more generally, any sequence ξ for which $\xi_1 = 0$.

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[1] Conventions vary about whether the sequences in ℓ^p begin with the zeroeth term or the first. For purposes of this problem, consider them as beginning with the first.

The argument for R is similar. If $p < \infty$ then

$$\|R\xi\|^p = \sum_{n=1}^{\infty} |(R\xi)_n|^p = 0 + \sum_{n=2}^{\infty} |\xi_{n-1}|^p = \sum_{n=1}^{\infty} |\xi_n|^p = \|\xi\|^p.$$

If $p = \infty$ then

$$\|R\xi\| = \sup_{n\geq 1} |(R\xi)_n| = \sup_{n\geq 2} |(R\xi)_n| = \sup_{n\geq 2} |\xi_{n-1}| = \sup_{n\geq 1} |\xi_n| = \|\xi\|.$$

The second inequality holds because $|(R\xi)_1| = 0$ is at least as small as the other elements in the sequence. In either case $\|R\xi\| = \|\xi\|$ and hence $\|R\xi\| \leq \|\xi\|$. R is therefore bounded and to show that $\|R\| = 1$ it suffices to consider *any* non-zero sequence ξ.

2. What is

   (a) L^m?
   Solution:

   $$(L^m\xi)_n = \xi_{n+m}$$

   (b) R^m?
   Solution:

   $$(R^m\xi)_n = \begin{cases} 0 & \text{if } n \leq m, \\ \xi_{n-m} & \text{if } n > m. \end{cases}$$

   (c) LR?
   Solution:

   $$(LR)_n = \xi_n$$

   (d) RL?
   Solution:

   $$(RL)_n = \begin{cases} 0 & \text{if } n = 1, \\ \xi_n & \text{if } n > 1. \end{cases}$$

3. Show that

   (a) L is surjective, but not injective.
   Solution:
   Since $LR\xi = \xi$ every $\xi \in \ell^p$ is L of something in ℓ^p, *i.e.* L is surjective. The sequence

   $$\xi_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

   is non-zero, but $L\xi = 0$, so L is not injective.

(b) R is injective, but not surjective.
Solution:
$LR\xi = \xi$, so $R\xi = 0$ implies $\xi = 0$. Thus R is injective. The particular sequence ξ defined in the previous part is not R of anything, so R is not surjective.

4. Prove that for any $\xi \in \ell^p$

$$\lim_{m\to\infty} L^m\xi = 0$$

if $1 \le p < \infty$.
Solution:

$$(L^m\xi)_n = \xi_{n+m}$$

so

$$\|L^m\xi\|^p = |\sum_{n=1}^{\infty} |\xi_{n+m}|^p = \sum_{n=m+1}^{\infty} |\xi_n|^p.$$

What we need to show is that for any $\epsilon > 0$ there is an M such that if $m \ge M$ then $\|L^m\xi\| < \epsilon$. By hypothesis, $\xi \in \ell^p$. In other words, the series

$$\sum_{n=1}^{\infty} |\xi_n|^p$$

converges. By definition the sequence of partial sums converges. In other words, there is, for each $\epsilon' > 0$, an M such that

$$\left|\sum_{n=1}^{\infty} |\xi_n|^p - \sum_{n=1}^{m} |\xi_n|^p\right| < \epsilon'$$

whenever $m \ge M$. Note that

$$\sum_{n=1}^{\infty} |\xi_n|^p - \sum_{n=1}^{m} |\xi_n|^p = \sum_{n=m+1}^{\infty} |\xi_n|^p.$$

This is clearly non-negative, so the outer absolute value signs in the preceding equation are irrelevant. Thus what we have is exactly what we need, provided we take $\epsilon' = \epsilon^p$.