MA 3421
Assignment 1
Due 10 October 2012

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1. Show that $B(T)$, the set of bounded functions on an arbitrary set T, with the usual norm
$$\|x\| = \sup_{t\in T} |x(t)|,$$
is a Banach space, and therefore that ℓ^∞ is a Banach space.
Solution: That $B(T)$ is a normed vector space is trivial to check. The main thing to check is completeness. Suppose then that x_n is a Cauchy sequence in $B(T)$, *i.e.* that for all $\epsilon > 0$ there is an n such that if $k, l \geq n$ then
$$\sup_{t\in T} |x_k(t) - x_l(t)| < \epsilon.$$
Since the supremum is an upper bound we then have
$$|x_k(t) - x_l(t)| < \epsilon$$
for each $t \in T$. Thus $x_n(t)$ is, for each $t \in T$, a Cauchy sequence in $\mathbf{R}$. Since $\mathbf{R}$ is complete there must be an $x(t)$ such that
$$\lim_{n\to\infty} x_n(t) = x(t).$$
So far we only have pointwise convergence. We need convergence in the $B(T)$ norm.

With ϵ, n as above we have, for each $t \in T$,
$$|x_k(t) - x_l(t)| < \epsilon,$$

as stated previously. Letting l tend to infinity, and using the continuity of addition of real numbers and of the absolute value function,

$$|x_k(t) - x(t)| < \epsilon.$$

Since this holds for all $t \in T$,

$$\sup_{t \in T} |x_k(t) - x(t)| < \epsilon,$$

i.e.,

$$\|x_k - x\| < \epsilon.$$

This holds for all $k \geq n$. Since there is such an n for each $\epsilon > 0$ the sequence x_n converges to x.

Finally, note that $\ell^\infty = B(\mathbf{Z})$, so ℓ^∞ is a Banach space.

2. Show that (c), the space of convergent sequences, with the usual norm

$$\|\xi\| = \sup_n |\xi_n|,$$

is a Banach space. The simplest way to do this is to use the result of the previous problem and the subspace criterion proved in lecture. *Solution:* As shown in first year, convergent sequences are bounded and linear combinations of convergent sequences are convergent, so (c) is a subspace of ℓ^∞. It was shown in class that a subspace of a Banach with the induced norm is a Banach space if and only if it is closed. (c) is a subspace of ℓ^∞ and its norm is the one induced from ℓ^∞, so it suffices now to show that (c) is a closed subspace.

First note that $\mathbf{R}$ is complete, so we can equally well think of (c) as the space of *Cauchy* sequences. A sequence ξ is *not* Cauchy if there is an $\epsilon > 0$ such that for all n there are $k, l > n$ for which

$$|\xi_k - \xi_l| \geq \epsilon.$$

In this case, if $\|\eta - \xi\| < \frac{\epsilon}{3}$ then, using the reverse triangle inequality,

$$|\xi_k - \eta_l| \geq |\xi_k - \xi_l| - |\xi_l - \eta_l| \geq \epsilon - \frac{\epsilon}{3} = 2\frac{\epsilon}{3}.$$

Using it again,

$$|\eta_k - \eta_l| \geq |\xi_k - \eta_l| - |\xi_k - \eta_k| \geq 2\frac{\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3}.$$

The sequence η thus also fails to be Cauchy. η was an arbitrary element in the ball of radius $\frac{\epsilon}{3}$ about ξ, so the whole ball belongs to $\ell^\infty - (c)$. ξ itself was an arbitrary element in $\ell^\infty - (c)$, so $\ell^\infty - (c)$ contains a ball about each of its points. In other words it is open. (c) must then be closed.

3. Show that in any normed space addition and scalar multiplication are continuous. Show also that the norm function and metric functions are continuous.
Solution: For continuity of addition we must check that for any x, y in our space and any $\epsilon > 0$ there are $\delta_x > 0$ and $\delta_y > 0$ such that if

$$\|x' - x\| < \delta_x$$

$$\|y' - y\| < \delta_x$$

then

$$\|(x' + y') - (x + y)\| < \epsilon.$$

This is true with $\delta_x = \delta_y = \frac{\epsilon}{2}$ by the triangle inequality, since

$$(x' + y') - (x + y) = (x' - x) + (y' - y).$$

For continuity of scalar multiplication we must show that for any scalar α, any vector x and any $\epsilon > 0$ there are $\delta_\alpha > 0$ and $\delta_x > 0$ such that if

$$|\alpha' - \alpha| < \delta_\alpha$$

and

$$\|x' - x\| < \delta_x$$

then

$$\|\alpha' x' - \alpha x\| < \epsilon.$$

The proof is similar to the previous one, but now we use the identity

$$\alpha' x' - \alpha x = \alpha'(x' - x) + (\alpha' - \alpha)x.$$

Choosing

$$\delta_\alpha = \frac{\epsilon}{2\|x\|}$$

and

$$\delta_x = \frac{\epsilon}{2(\alpha + \delta_\alpha)}$$

works. If $x = 0$ then δ_α can be chosen arbitrarily and δ_x as above.

For continuity of the norm we must show that for every vector x and every $\epsilon > 0$ there is a δ_x such that if

$$\|x' - x\| < \delta_x$$

then

$$|\|x'\| - \|x\|| < \epsilon.$$

The choice $\delta_x = \epsilon$ works, by the reverse triangle inequality.

The continuity of the metric can be shown by an argument similar to the previous three, but this is unnecessary because

$$d(x, y) = \|x + (-1)y\|,$$

so the metric function is a composition of scalar multiplication, addition and the norm function. Since these are all continuous the composite function must be continuous as well.