MA 3421
Assignment 4
Due 14 November 2012

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1. A *sesquilear form* on an inner product space E is, by definition, a function $s\colon E \times E \to \mathbf{K}$ satisfying
$$s(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 s(x_1, y) + \alpha_2 s(x_2, y)$$
and
$$s(x, \beta_1 y_1 + \beta_2 y_2) = \overline{\beta}_1 (x, y_1) + \overline{\beta}_2 (x, y_2).$$
Here $\mathbf{K}$ is $\mathbf{R}$ or $\mathbf{C}$, depending on whether E is a real or complex inner product space. In the former case the complex conjugation is, of course, irrelevant. A sesquilinear form s is called *continuous* if $x_n \to x$ and $y_n \to y$ imply $s(x_n, y_n) \to s(x, y)$. A sesquilinear form s is called bounded when there is a $\lambda \geq 0$ such that for all x and y one has $|s(x, y)| \leq \lambda \|x\| \|y\|$. Prove that a sesquilinear form is continuous if and only if it is bounded.

2. Note that if A is a linear transformation from E to E and s is defined by
$$s(x, y) = (Ax|y)$$
then s is a sequilinear form on E. Prove that s is continuous if and only if A is.

3. Suppose E is a Hilbert space and s is a continuous sesquilinear form. Prove that there is one and only one continuous linear transformation $A\colon E \to E$ such that
$$s(x, y) = (Ax|y).$$