1. (a) $a_n = (-1)n$.
   (b) $f(z) = \sin(z)$.
   (c) $f(z) = \sin(z)$, $g(z) = \cos(z)$.
   (d) $U = \mathbf{C} - \{0\}$.
   (e) $w = 0$, $U = \mathbf{C}$, $\gamma(t) = e^{it}$.
   (f) Drawn in lecture.
   (g) $U = \mathbf{C} - \{0\}$, $f(z) = 1/z$.
   (h) Impossible, since zeroes of holomorphic functions are isolated.
   (i) $w = 0$, $f(z) = 1/z^2$.
   (j) $w = 0$, $f(z) = \exp(1/z)$.

2. (a) $c_l = \sum_{j=0}^{l} a_j b_{l-j}$.
   (b) The radius of convergence of the product is at least as large as the minimum of the radii of convergence of the multiplicands.
   (c)

$$f(z) = \sqrt{1 - 2xz + z^2} = \sum_{j=0}^{\infty} a_j z^j.$$

Take $b_j = a_j$. Then

$$\sum_{l=0}^{\infty} c_l z^l = 1 - 2xz + z^2,$$

so

$$c_0 = 1 \qquad c_1 = -2x \qquad c_2 = 1$$

and all other c's are zero.

$$a_0 = f(0) = 1.$$

The other a's are obtained from the formula $c_l = \sum_{j=0}^{l} a_j b_{l-j} = sum_{j=0}^{l} a_j a_{l-j}$. For $l = 1$

$$a_0 a_1 + a_1 a_0 = c_1 = -2x$$

or

$$2a_1 = -2x$$

so $a_1 = -x$. Then

$$a_0 a_2 + a_1 a_1 + a_2 a_0 = c_2 = 1$$

or

$$2a_2 + x^2 = 1$$

so

$$a_2 = \frac{1}{2}(1 - x^2).$$

Then

$$a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0 = c_3 = 0$$

or

$$2a_3 - 2x\frac{1}{2}(1 - x^2) = 0$$

so

$$a_3 = \frac{1}{2}(x - x^3).$$

We could continue indefinitely, but are only asked for the a's up through a_3.

3. (a) There aren't any.
   (b) $w = 0$, $f(z) = \exp(z)$.
   (c) Set $g(z) = (f(z) - w)^{-1}$. If there is a z such that $f(z) = w$ then the sequence $z_n = z$ satisfies $\lim_{n\to\infty} f(z_n) = f(z) = w$ and we are done. Otherwise g is non-constand and holomorphic in $\mathbf{C}$, hence unbounded. Since g is unbounded there is, for each non-negative integer n, a z_n such that $|g(z)| > n$, *i.e.* $f(z_n) - w| < 1/n$. But then $\lim_{n\to\infty} f(z_n) = w$.

4. (a) This is a special case of the problem in the next part.
   (b) Let

$$f(z) = \frac{\exp(2\pi i\xi z)}{1 + z + z^2}.$$

We integrate over a rectangular path γ with vertices at $-R$, R, $R \pm iR$ and $-R \pm iR$, visited in that order. The sign in $\pm$ is the sign of ξ. If $\xi = 0$ then wither sign will work. With this choice, $|\exp(2\pi i\xi z)| \leq 1$ on the whole path. By the Cauchy integral formula

$$\int_\gamma f(z)\, dz = 2\pi i n(\gamma, w) \operatorname*{Res}_{z=w} f(z)$$

where w is the pole of f inside the rectangle, $w = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. From the picture $n(\gamma, w) = \pm 1$. By the multiplication formula,

$$\operatorname*{Res}_{z=w} f(z) = \exp(2\pi i\xi w) \operatorname*{Res}_{z=w} \frac{1}{1 + z + z^2},$$

By the limit formula,

$$\operatorname*{Res}_{z=w} \frac{1}{\operatorname{Res}_{z=w}} = \frac{1}{\pm i\sqrt{3}}$$

so

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi i n(\gamma, w)\operatorname*{Res}_{z=w} f(z)\\ &= \frac{2\pi}{\sqrt{3}}\exp\left(\pi i\xi(-1 \pm i\sqrt{3})\right)\\ &= \frac{2\pi}{\sqrt{3}}\exp(-\pi\sqrt{3}|\xi|)\exp(-\pi i\xi).\end{aligned}$$

Now

$$\int_\gamma f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\tilde{\gamma}} f(z)\,dz$$

where $\tilde{\gamma}$ is the portion of γ which goes from R to $-R$ via $R \pm iR$ and $-R \pm iR$. As $R \to \infty$,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

while

$$|\int_{\tilde{\gamma}} f(z)\,dz| \le 4R \max_{[\tilde{\gamma}]}|f| \le \frac{4R}{R^2 - R - 1}$$

so

$$\lim_{R\to\infty}\int_{\tilde{\gamma}} f(z)\,dz = 0.$$

Therefore

$$\int_{-\infty}^{\infty}\frac{\exp(2\pi i\xi x)}{1 + x + x^2}\,dx = \frac{2\pi}{\sqrt{3}}\exp(-\pi\sqrt{3}|\xi|)\exp(-\pi i\xi).$$