1. (20 points) For each of the following, either give an example or a brief explanation of why it is impossible. (4 points each)

   (a) A power series which converges everywhere, but whose sum is not a bounded function.
   Solution: Many examples. One is $\sum_{n=0}^{\infty} z^n/n!$.

   (b) A closed path γ in $\mathbf{C} - \{0\}$ which is not contractible.
   Solution: Many examples. One is $\gamma(t) = \exp(it)$, $-\pi \le t \le \pi$.

   (c) A function f, analytic on $\mathbf{C}$ except for poles at the points $z = n$, $n \in \mathbf{Z}$.
   Solution: Many examples. One is $f(z) = \cot(\pi z)$

   (d) A function f on $\mathbf{C}$ which is continuous but not differentiable.
   Solution: Many examples. One is $f(z) = |z|$.

   (e) An open set U and a function f, holomorphic in U, such that there is no function F, defined in U, with $F' = f$ throughout U.
   Solution: Many examples. One is $U = \mathbf{C} - \{0\}$, $f(z) = 1/z$.

2. (20 points)

   (a) (5 points) State Liouville's Theorem.
   Solution: Any bounded analytic function on $\mathbf{C}$ is constant.

   (b) (15 points) Suppose that f is analytic in $\mathbf{C}$ and satisfies $f(z+m+in) = f(z)$ for all $m, n \in \mathbf{Z}$. Prove the f is constant.
   Solution: Every $z \in \mathbf{C}$ is of the form $x+iy+m+in$ for some integers m and n and some real numbers $0 \le x, y \le 1$. By hypothesis then $f(z) = f(x+iy)$, so

   $$|f(z)| \le \max_{(x,y)\in R} |f(x+iy)|$$

   where R is the unit square $0 \le x, y \le 1$. This maximum exists because $|f(x+iy)|$ is a continuous function of x and y, and continuous real valued functions on a product of intervals have a maximum (and minimum, but we don't care). So f must be constant by Liouville's theorem.

3. (20 points) Suppose that

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

for all $z \in \mathbf{C}$.

(a) (2 points) Find the power series expansion for f'.
Solution:

$$f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n.$$

(b) (2 points) Where does it converge?
Solution: Everywhere.

(c) (2 points) Find the power series expansion for f^2.
Solution: By the rule for multiplication of power series

$$f(z)^2 = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j a_{n-j} \right) z^n.$$

(d) (2 points) Where does it converge?
Solution: Everywhere.

(e) (12 points) Suppose that

$$f'(x)^2 + f(x)^2 = 1, \qquad f(0) = 0, \qquad f'(0) = 1.$$

Find a_0, a_1, a_2, a_3, a_4 and a_5
Solution:

$$\begin{aligned} f(z)^2 = a_0^2 + 2a_0a_1 z + (2a_0a_2 + a_1^2)z^2 + (2a_0a_3 + 2a_1a_2)z^3 \\ + (2a_0a_4 + 2a_1a_3 + a_2^2)z^4 + \cdots, \end{aligned}$$

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots$$

and

$$\begin{aligned} f'(z)^2 = a_1^2 + 4a_1a_2 z + (6a_1a_3 + 4a_2^2)z^2 + (8a_1a_4 + 12a_2a_3)z^3 \\ + (10a_1a_5 + 16a_2a_4 + 9a_3^2)z^4 + \cdots. \end{aligned}$$

So

$$\begin{aligned} a_0^2 + a_1^2 &= 1, \\ 2a_0a_1 + 4a_1a_2 &= 0, \\ 2a_0a_2 + a_1^2 + 6a_1a_3 + 4a_2^2 &= 0, \\ 2a_0a_3 + 2a_1a_2 + 8a_1a_4 + 12a_2a_3 &= 0, \end{aligned}$$

and

$$2a_0a_4 + 2a_1a_3 + a_2^2 + 10a_1a_5 + 16a_2a_4 + 9a_3^2 = 0.$$

By assumption, $a_0 = f(0) = 0$ and $a_1 = f'(0) = 1$. Solving the equations above for the next few coefficients,

$$a_2 = 0, \qquad a_3 = -1/6, \qquad a_4 = 0, \qquad a_5 = 1/120.$$

4. (20 points)

(a) (4 points) Write

$$\int_{-\pi}^{\pi} \frac{d\theta}{a + \cos\theta}$$

for $a > 1$ as

$$\int_\gamma f(z)\,dz$$

where f is a rational function and $\gamma(t) = e^{it}$ for $-\pi \le t \le \pi$.
Solution:

$$f(z) = \frac{-i}{(\frac{1}{2}z^2 + az + \frac{1}{2})}.$$

(b) (3 points) Find the poles of f. If you didn't do part (a) then find the poles of the function

$$g(z) = (z^4 - 6z^2 + 1)^{-1}$$

instead.
Solution: The poles of f are at

$$z_+ = -a + \sqrt{a^2 - 1}, \qquad z_- = -a - \sqrt{a^2 - 1}.$$

(c) (2 points) Find the orders of the poles of f, or of g if you didn't do part (a).
Solution: These are simple poles. You would have a double pole if $a = \pm 1$, but we are told that $a > 1$.

(d) (4 points) Find the residues of the poles of f, or of g if you didn't do part (a).
Solution:

$$f(z) = \frac{-2i}{(z - z_+)(z - z_-)},$$

so

$$\operatorname*{Res}_{z=z_+} f(z) = \lim_{z\to z_+} (z-z_+)f(z) = \lim_{z\to z_+} \frac{-2i}{z-z_-} = \frac{-2i}{z_+-z_-} = \frac{-i}{\sqrt{a^2-1}}.$$

Similarly

$$\operatorname*{Res}_{z=z_-} f(z) = \frac{i}{\sqrt{a^2-1}}.$$

(e) (2 points) Find the winding number of γ about the poles of f, or of g if you didn't do part (a).
Solution: $z_+z_- = 1$ and hence, $|z_+||z_-| = 1$ so one of the poles is inside. Since $|z_-| > |z_+|$ it must be z_+ which is inside. Thus $n(\gamma, z_+) = 1$ and $n(\gamma, z_-) = 0$.

(f) (5 points) Evaluate

$$\int_{-\pi}^{\pi} \frac{d\theta}{a+\cos\theta}.$$

Solution:

$$\int_{-\pi}^{\pi} \frac{d\theta}{a+\cos\theta} = \int_\gamma f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_+} f(z) = \frac{2\pi}{\sqrt{a^2-1}}.$$

5. (20 points) Suppose $w \in \mathbf{C}$ is not an integer. Let

$$f(z) = \pi\cot(\pi z)(z-w)^{-2}$$

Let γ_N be a path going around the square with corners at $\pm(N+\frac{1}{2}) \pm i(N+\frac{1}{2})$ in the counterclockwise direction.

(a) (3 points) Find the poles of f.
Solution: There are poles at w, coming from the factor $(z-w)^2$, and at the integers, coming from the factor $\cot(\pi z)$.

(b) (2 points) Find their orders.
Solution: The pole at w is of order 2, while the poles at the integers are of order 1.

(c) (3 points) Find their residues.
Solution: For the poles at the integers, which are simple, we can use the multiplication formula for residues. $\pi\cot(\pi z)$ has a pole of residue 1 at n, so f has a pole of residue $(n-w)^2$ there. For

the pole at w we compute power series, or at least the first few terms:

$$\begin{aligned}\cot(\pi z) &= \cot(\pi w) + \pi \cot'(\pi w)(z-w) + \cdots \\ &= \cot(\pi w) - \pi \csc^2(\pi w)(z-w) + \cdots\end{aligned}$$

so

$$f(z) = \pi \cot(\pi w)(z-w)^{-2} - \pi^2 \csc^2(\pi w)(z-w)^{-1} + \cdots.$$

$\mathrm{Res}_{z=w} f(z)$ is the coefficient of $(z-w)^{-1}$, in other words, $-\pi^2 \csc^2(\pi w)$

(d) (2 points) Find the index of γ_N about these poles.
Solution: The index, or winding number, about n is 1, for $-N \leq n \leq N$ and zero for the other integers. The index about w is also 1, once N is large enough.

(e) (4 points) Prove that

$$\lim_{N\to\infty} \int_{\gamma_N} f(z)\, dz = 0.$$

Solution: We only care about large N, so assume $|w| \leq N + \frac{1}{2}$. We know that $\pi \cot(\pi z)$ is bounded on $[\gamma_N]$, with a bound which is independent of N. In fact this bound is $\coth \frac{\pi}{2}$, but the exact value is not important, so just call it K. Then $|f(z)|$ is bounded by $K/(N + \frac{1}{2} - |w|)^2$ on $[\gamma_N]$, and hence $\left|\int_{\gamma_N} f(z)\, dz\right|$ is bounded by $K(8N+4)/(N + \frac{1}{2} - |w|)^2$. But this tends to zero as $N \to \infty$.

(f) (6 points) Evaluate

$$\sum_{n=-\infty}^{\infty} (w-n)^{-2}.$$

Solution: By Cauchy's Theorem,

$$\begin{aligned}\textstyle\int_{\gamma_N} f(z)\, dz &= 2\pi i \textstyle\sum_a n(\gamma_N, a)\, \mathrm{Res}_{z=a} f(z) \\ &= 2\pi i \left(\textstyle\sum_{n=-N}^{N} (w-n)^2 - \pi^2 \csc^2(\pi w)\right).\end{aligned}$$

We've already seen that the limit is zero, so

$$\lim_{N\to\infty} \sum_{n=-N}^{N} (w-n)^{-2} = \pi^2 \csc^2(\pi w)$$