UNIVERSITY OF DUBLIN
TRINITY COLLEGE

Faculty of Science

school of mathematics

Scholarship Exam 2010

Course 2325

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ATTEMPT SIX QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. (20 points) For each of the following, either give an example or a brief explanation of why it is impossible. (2 points each)

   (a) A series which converges, but not absolutely.

   (b) A power series which converges only at one point.

   (c) Holomorphic functions f and g whose power series about 0 converge everywhere, $g(0) \neq 0$, and the power series for their product $f \cdot g$ *does not* converge everywhere.

   (d) A continous complex valued function defined on $\mathbf{C}$ which is *not* holomorphic.

   (e) An open set U, a holomorphic function f on U, and a piecewise continuously differentiable path γ in U such that $\int_\gamma f(z)\,dz \neq 0$.

   (f) A point $w \in \mathbf{C}$ and a function f, holomorphic in $\mathbf{C} - \{w\}$, with a simple pole at w of residue 0.

   (g) Complex numbers w and z such that $\sqrt{wz} \neq \sqrt{w}\sqrt{z}$.

   (h) Complex numbers w and z with $w \neq z$ and $\exp(w) = \exp(z)$.

   (i) A holomorphic function f on $\mathbf{C} - \{0\}$ with a removable singularity at 0.

   (j) A non-constant periodic holomorphic function on $\mathbf{C}$.

2. (20 points)

   (a) (4 points) If

   $$f(z) = \sum_{j=0}^{\infty} a_j (z - w)^j$$

   and

   $$f'(z) = \sum_{j=0}^{\infty} c_j (z - w)^j$$

   then what is the relation between the c's and the a's?

   (b) (4 points) What can you say about the radii of convergence?

   (c) (12 points) From the fact that the tangent function satisfies the relation

   $$\tan'(z) = 1 + \tan(z)^2,$$

   compute the power series expansion of $\tan(z)$ up through the z^5 term.

3. (20 points)

   (a) (4 points) What is the Cauchy Integral Formula?

   (b) (4 points) Under what conditions on the function and the coutour is this formula known to hold?

   (c) (12 points) Suppose f is a holomorphic function on $U = \mathbf{C} - \{0\}$ and that

   $$|f(z) \leq |z|^{1/2}$$

   for all $z \in U$. Show that f is everywhere zero.

4. (20 points) Evaluate the following by contour integration. Be sure to justify your calculations.

   (a) (8 points)

$$\int_{-\infty} \frac{dx}{1+x^4}$$

   (b) (12 points)

$$\int_{-\infty} \frac{\exp(2\pi i \xi x)dx}{1+x^4}$$

   where $\xi \in \mathbf{R}$.