UNIVERSITY OF DUBLIN
TRINITY COLLEGE

Faculty of Science

school of mathematics

Scholarship Exam 2010

Course 2325

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ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. (20 points) For each of the following, either give an example or a brief explanation of why it is impossible. (2 points each)

   (a) A bounded, but not convergent, sequence of complex numbers.

   (b) A non-zero function holomorphic f whose power series expansion about 0 has infinitely many zero coefficients.

   (c) Holomorphic functions f and g whose power series about 0 converge everywhere, $g(0) \neq 0$, and the power series for f/g *does not* converge everywhere.

   (d) An open set U which is *not* star-shaped.

   (e) A point w in a star-shaped region U and a closed path γ in U such that $n(\gamma, w) \neq 0$.

   (f) A *non-contractible* closed path γ in $\mathbf{C} - \{-1, 1\}$ such that $n(\gamma, -1) = n(\gamma, 1) = 0$. *Note:* Pictures will be considered acceptable, if clearly drawn.

   (g) An open set U and a function f, holomorphic in U, such that there is *no* function F, defined in U, with $F' = f$ throughout U.

   (h) A non-zero function f, holomorphic on $\mathbf{C}$, with $f(z) = 0$ for all real z.

   (i) A point $w \in \mathbf{C}$ and a function f, holomorphic in $\mathbf{C} - \{w\}$, with a pole at w of residue 0.

   (j) A point $w \in \mathbf{C}$ and a function holomorphic in $\mathbf{C} - \{w\}$ with an essential singularity at w.

2. (20 points)

(a) (4 points) If

$$\sum_{j=0}^{\infty} a_j (z-w)^j \sum_{k=0}^{\infty} b_k (z-w)^k = \sum_{l=0} c_l (z-w)^l$$

then what is the relation between the c's and the a's and b's?

(b) (4 points) What can you say about the radii of convergence?

(c) (12 points) Find the power series, up through the z^3 term, for the function

$$f(z) = \sqrt{1 - 2xz + z^2}.$$

More precisely, assume that there is a holomorphic function f, defined for $|z| < r$ with some $r > 0$, such that

$$f(z)^2 = 1 - 2xz + z^2, \qquad f(0) = 1$$

and

$$f(z) = \sum_{j=0}^{\infty} a_j z^j.$$

Find a_0, a_1, a_2 and a_3. Your answer will, of course, depend on x, which should be considered an arbitrary complex number.

3. (20 points)

   (a) (4 points) What can you say about non-constant bounded holomorphic functions on $\mathbf{C}$?

   (b) (6 points) Give an example of a non-constant holomorphic function on $\mathbf{C}$ and a complex number w such that there is *no* $z \in \mathbf{C}$ for which $f(z) = w$.

   (c) (10 points) Prove that for every non-constant holomorphic function f on $\mathbf{C}$ and every complex number w, there is a sequence $z_0, z_1, z_2, \ldots$ such that

$$\lim_{n\to\infty} f(z_n) = w.$$

4. (20 points) Evaluate the following by contour integration. Be sure to justify your calculations.

   (a) (8 points)

$$\int_{-\infty} \frac{dx}{1+x+x^2}$$

   (b) (12 points)

$$\int_{-\infty} \frac{\exp(2\pi i\xi x)dx}{1+x+x^2}$$

   where $\xi \in \mathbf{R}$.