UNIVERSITY OF DUBLIN
TRINITY COLLEGE

Faculty of Science

school of mathematics

Annual Exam 2010

Course 2325

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ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. (20 points) For each of the following, either give an example or a brief explanation of why it is impossible. (4 points each)

   (a) A power series which converges everywhere, but whose sum is not a bounded function.

   (b) A closed path γ in $\mathbf{C} - \{0\}$ which is not contractible.

   (c) A function f, analytic on $\mathbf{C}$ except for poles at the points $z = n$, $n \in \mathbf{Z}$.

   (d) A function f on $\mathbf{C}$ which is continuous but not differentiable.

   (e) An open set U and a function f, holomorphic in U, such that there is no function F, defined in U, with $F' = f$ throughout U.

2. (20 points)

   (a) (5 points) State Liouville's Theorem.

   (b) (15 points) Suppose that f is analytic in $\mathbf{C}$ and satisfies $f(z + m + in) = f(z)$ for all $m, n \in \mathbf{Z}$. Prove the f is constant.

3. (20 points) Suppose that

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

for all $z \in \mathbf{C}$.

(a) (2 points) Find the power series expansion for f'.

(b) (2 points) Where does it converge?

(c) (2 points) Find the power series expansion for f^2.

(d) (2 points) Where does it converge?

(e) (12 points) Suppose that

$$f'(x)^2 + f(x)^2 = 1, \qquad f(0) = 0, \qquad f'(0) = 1.$$

Find a_0, a_1, a_2, a_3, a_4 and a_5

4. (20 points)

   (a) (4 points) Write

   $$\int_{-\pi}^{\pi} \frac{d\theta}{a + \cos\theta}$$

   for $a > 1$ as

   $$\int_{\gamma} f(z)\, dz$$

   where f is a rational function and $\gamma(t) = e^{it}$ for $-\pi \leq t \leq \pi$.

   (b) (3 points) Find the poles of f. If you didn't do part (a) then find the poles of the function

   $$g(z) = (z^4 - 6z^2 + 1)^{-1}$$

   instead.

   (c) (2 points) Find the orders of the poles of f, or of g if you didn't do part (a).

   (d) (4 points) Find the residues of the poles of f, or of g if you didn't do part (a).

   (e) (2 points) Find the winding number of γ about the poles of f, or of g if you didn't do part (a).

   (f) (5 points) Evaluate

   $$\int_{-\pi}^{\pi} \frac{d\theta}{a + \cos\theta}.$$

5. (20 points) Suppose $w \in \mathbf{C}$ is not an integer. Let

$$f(z) = \pi \cot(\pi z)(z - w)^{-2}$$

Let γ_N be a path going around the square with corners at $\pm(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$ in the counterclockwise direction.

(a) (3 points) Find the poles of f.

(b) (2 points) Find their orders.

(c) (3 points) Find their residues.

(d) (2 points) Find the index of γ_N about these poles.

(e) (4 points) Prove that

$$\lim_{N\to\infty} \int_{\gamma_N} f(z)\, dz = 0.$$

(f) (6 points) Evaluate

$$\sum_{n=-\infty}^{\infty} (w - n)^{-2}.$$