MA 2325
Assignment 6
Due 16 December 2009

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1. Evaluate
$$\int_{-\pi}^{\pi} \frac{A\cos\theta + B\sin\theta + C}{a\cos\theta + b\sin\theta + c}\, d\theta$$
by contour integration, where $a^2 + b^2 < c^2$.
Solution: First note that $c \neq 0$, so $c < 0$ or $c > 0$. There is no need to consider $c < 0$, since the integral is obviously unchanged by switching the signs of A, B, C, a, b and c.

In lecture we computed
$$\int_{-\pi}^{\pi} \frac{1}{a\cos\theta + b\sin\theta + c}\, d\theta = \frac{2\pi}{\sqrt{c^2 - a^2 - b^2}}$$
by integrating $\frac{2i}{(a-ib)z^2+2cz+(a+ib)}$ over the path
$$\gamma(\theta) = e^{i\theta} \qquad -\pi \leq \theta \leq \pi.$$
The most straightforward approach would be to solve the present problem by the same method, *i.e.* to choose f so that the desired integral is $\int f(z)\, dz$:
$$f(z) = -\frac{i}{z}\frac{(A - iB)z^2 + 2Cz + (A - iB)}{(a - ib)z^2 + 2cz + (a + ib)}.$$
This will work, but it is unnecessarily complicated. In addition to having six parameters to deal with, the integrand f has three poles, of which two within the unit circle.

It is possible to simplify the argument somewhat by forgetting arbitrary A, B and C for the moment and concentrating on the special case $A = 1$, $B = i$ and $C = 0$. For this we need the integrand

$$f(z) = \frac{-2iz}{(a-ib)z^2 + 2cz + (a+ib)}.$$

With this choice of f and γ,

$$\gamma'(\theta) = ie^{i\theta} = i(\cos\theta + i\sin\theta)$$

and

$$\begin{aligned} f(\gamma(\theta)) &= \frac{-2ie^{i\theta}}{(a-ib)e^{2i\theta} + 2ce^{i\theta} + (a+ib)} \\ &= \frac{-2i}{(a-ib)e^{i\theta} + 2c + (a+ib)e^{-i\theta}} \\ &= \frac{-2i}{a(e^{i\theta} + e^{-i\theta}) + 2c + b(-ie^{i\theta} + ie^{-i\theta})} \\ &= \frac{-i}{a\cos\theta + b\sin\theta + c} \end{aligned}$$

so

$$f(\gamma(\theta))\gamma'(\theta) = \frac{\cos\theta + i\sin\theta}{a\cos\theta + b\sin\theta + c}$$

and

$$\int_\gamma f(z)\,dz = \int_{-\pi}^{\pi} \frac{\cos\theta + i\sin\theta}{a\cos\theta + b\sin\theta + c}\,d\theta.$$

Of course, f was chosen specifically to make this true. By the Cauchy Residue Theorem,

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname*{Res}_{z=w} f(z)$$

where w is the pole of f inside the unit circle. The path gamma has winding number 0 about the other pole, so that pole has no contribution to the sum. As shown in lecture,

$$w = \frac{-c + \sqrt{c^2 - a^2 - b^2}}{a - ib}$$

and

$$\operatorname*{Res}_{z=w} \frac{2}{(a-ib)z^2 + 2cz + (a+ib)} = -\frac{1}{\sqrt{c^2 - a^2 - b^2}}.$$

From the multiplication theorem for simple poles,

$$\operatorname*{Res}_{z=w} f(z) = \frac{iw}{\sqrt{c^2 - a^2 - b^2}}$$

and hence

$$\int_{-\pi}^{\pi} \frac{\cos\theta + i\sin\theta}{a\cos\theta + b\sin\theta + c}\, d\theta = \frac{2\pi w}{\sqrt{c^2 - a^2 - b^2}}.$$

A little algebra shows that

$$w = -\frac{a + ib}{c + \sqrt{c^2 - a^2 - b^2}}$$

so

$$\int_{-\pi}^{\pi} \frac{\cos\theta + i\sin\theta}{a\cos\theta + b\sin\theta + c}\, d\theta = -\frac{2\pi(a + ib)}{\sqrt{c^2 - a^2 - b^2}(c + \sqrt{c^2 - a^2 - b^2})}.$$

The real part of the integral is the integral of the real part, and similarly for the imaginary part, so

$$\int_{-\pi}^{\pi} \frac{\cos\theta}{a\cos\theta + b\sin\theta + c}\, d\theta = -\frac{2\pi a}{\sqrt{c^2 - a^2 - b^2}(c + \sqrt{c^2 - a^2 - b^2})}$$

and

$$\int_{-\pi}^{\pi} \frac{\sin\theta}{a\cos\theta + b\sin\theta + c}\, d\theta = -\frac{2\pi b}{\sqrt{c^2 - a^2 - b^2}(c + \sqrt{c^2 - a^2 - b^2})}.$$

We already saw that

$$\int_{-\pi}^{\pi} \frac{1}{a\cos\theta + b\sin\theta + c}\, d\theta = \frac{2\pi}{\sqrt{c^2 - a^2 - b^2}}$$

By linearity,

$$\int_{-\pi}^{\pi} \frac{A\cos\theta + B\sin\theta + C}{a\cos\theta + b\sin\theta + c}\, d\theta = 2\pi\frac{-Aa - Bb + C(c + \sqrt{c^2 - a^2 - b^2})}{\sqrt{c^2 - a^2 - b^2}(c + \sqrt{c^2 - a^2 - b^2})}.$$

Another way to write this is

$$\int_{-\pi}^{\pi} \frac{A\cos\theta + B\sin\theta + C}{a\cos\theta + b\sin\theta + c}\, d\theta = 2\pi\frac{Cc - Aa - Bb + Cc\sqrt{1 - a^2/c^2 - b^2/c^2}}{c^2 - a^2 - b^2 + c^2\sqrt{1 - a^2/c^2 - b^2/c^2})}.$$

This form has the advantage that it works for $c < 0$ as well, since it is unchanged on switching the signs of A, B, C, a, b and c. A useful check on this result is the fact that we get the correct answer, 2π, when $A = a$, $B = b$ and $C = c$.

2. Evaluate
$$\sum_{-\infty}^{\infty} \frac{1}{1+n^2}$$
by contour integration.
Hint: Consider the integral of the function
$$f(z) = \pi \cot(\pi z)(1+z^2)^{-1}.$$
on a square centred at 0 with side length an odd integer.
Solution: Since $\pi \cot(\pi z)$ has simple poles of residue 1 at the integers, f has simple poles at n with residue $(n^2+1)^{-1}$, by the division formula for residues. The other two poles are at the zeroes of z^2+1, *i.e.* at $\pm i$. By the limit formula,
$$\operatorname*{Res}_{z=\pm i}(z^2+1)^{-1} = \lim_{z\to\pm i}(z-\pm i)(z^2+1)^{-1} = \mp i2.$$
By the multiplication formula,
$$\operatorname*{Res}_{z=\pm i} f(z) = \pi \cot(\pi \pm i) \operatorname*{Res}_{z=\pm i}(z^2+1)^{-1} = -\frac{\pi}{2}\coth \pi$$
By the Cauchy Residue formula,
$$\int_{\gamma_N} f(z)\, dz = 2\pi i \sum \operatorname{Res} z = w f(z)$$
where γ_N, is the square contour of side length $2N+1$ referred to in the hint and the sum is over all poles inside this square. From the residues already computed,
$$\int_{\gamma_N} f(z)\, dz = 2\pi i \left(\sum_{-N}^{N} \frac{1}{n^2+1} - \pi \coth \pi \right)$$
or
$$\sum_{-N}^{N} \frac{1}{n^2+1} = \pi \coth \pi + \frac{1}{2\pi i}\int_{\gamma_N} f(z)\, dz.$$
On $[\gamma_N]$,
$$(z^2+1)^{-1} \le ((N+\frac{1}{2})^2-1)^{-1}$$
and
$$\pi \cot(\pi z) \le \pi \coth(3\pi/2)$$
so
$$\max_{[\gamma_N]} \le pi \coth(3\pi/2)((N+\frac{1}{2})^2-1)^{-1}.$$

since the contour is of length $4N + 2$,

$$\left|\frac{1}{2\pi i}\int_{\gamma_N} f(z)\,dz\right| \leq \coth(3\pi/2)\frac{2N+1}{(N+\frac{1}{2})^2 - 1}$$

so

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\gamma_N} f(z)\,dz = 0$$

and

$$\sum_{-\infty}^{\infty}\frac{1}{n^2+1} = \pi\coth\pi.$$

3. Show that the function

$$u(x,y) = \sum_{j=0}^{n/2}(-1)^j\frac{n!}{(2j)!(n-2j)!}x^{n-2j}y^{2j}$$

is harmonic. By a theorem proved in class, there is a holomorphic function f such that $u(x,y)$ is the real part of $f(x+iy)$. Find f. *Solution:* Repeated differentiation shows that

$$\frac{\partial u}{\partial x}(x,y) = \sum_{j=0}^{(n-1)/2}(-1)^j\frac{n!}{(2j)!(n-2j-1)!}x^{n-2j-1}y^{2j},$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \sum_{j=0}^{(n-2)/2}(-1)^j\frac{n!}{(2j)!(n-2j-2)!}x^{n-2j-2}y^{2j},$$

$$\frac{\partial u}{\partial y}(x,y) = \sum_{j=1}^{n/2}(-1)^j\frac{n!}{(2j-1)!(n-2j)!}x^{n-2j}y^{2j-1}$$

and

$$\frac{\partial u^2}{\partial y^2}(x,y) = \sum_{j=1}^{n/2}(-1)^j\frac{n!}{(2j-2)!(n-2j)!}x^{n-2j}y^{2j-2}.$$

We now change the index of summation, replacing j by k in the sum for $\partial^2 u/\partial x^2$ and by $k+1$ in the sum for $\partial^2 u/\partial y^2$.

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \sum_{k=0}^{(n-2)/2}(-1)^k\frac{n!}{(2k)!(n-2k-2)!}x^{n-2k-2}y^{2k},$$

and

$$\frac{\partial u^2}{\partial y^2}(x,y) = -\sum_{k=0}^{(n-2)/2}(-1)^k\frac{n!}{(2k)!(n-2k-2)!}x^{n-2k-2}y^{2k},$$

so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2}(x, y) = 0.$$

It's possible to find f by guessing. A more systematic method is to find v from the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

so

$$\frac{\partial v}{\partial y} = \sum_{j=0}^{(n-1)/2} (-1)^j \frac{n!}{(2j)!(n-2j-1)!} x^{n-2j-1} y^{2j}.$$

This can be true only if

$$v(x, y) = \sum_{j=0}^{(n-1)/2} (-1)^j \frac{n!}{(2j+1)!(n-2j-1)!} x^{n-2j-1} y^{2j+1} + \varphi(x).$$

for some function ϖ of one variable. Then

$$\frac{\partial v}{\partial x} = \sum_{j=0}^{(n-2)/2} (-1)^j \frac{n!}{(2j+1)!(n-2j-2)!} x^{n-2j-2} y^{2j+1} + \varphi'(x).$$

But

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\sum_{j=1}^{n/2} (-1)^j \frac{n!}{(2j-1)!(n-2j)!} x^{n-2j} y^{2j-1}.$$

Changing the index of summation as before,

$$\frac{\partial v}{\partial x} = -\sum_{k=0}^{(n-2)/2} (-1)^k \frac{n!}{(2k+1)!(n-2k-2)!} x^{n-2k-2} y^{2k+1}.$$

Comparing the two, we see that $\varphi'(x) = 0$, so φ is constant. Thus

$$v(x, y) = \sum_{j=0}^{(n-1)/2} (-1)^j \frac{n!}{(2j+1)!(n-2j-1)!} x^{n-2j-1} y^{2j+1} + \varphi.$$

Then

$$\begin{aligned} f(x+iy) &= u(x, y) + iv(x, y) \\ &= \sum_{j=0}^{n/2} (-1)^j \frac{n!}{(2j)!(n-2j)!} x^{n-2j} y^{2j} \\ &\quad + i \sum_{j=0}^{(n-1)/2} (-1)^j \frac{n!}{(2j+1)!(n-2j-1)!} x^{n-2j-1} y^{2j+1} + \varphi. \end{aligned}$$

This is correct, but not very enlightening. It becomes a bit clearer if we write $(-1)^j$ as i^{2j},

$$f(x+iy) = \sum_{j=0}^{n/2} \frac{n!}{(2j)!(n-2j)!} x^{n-2j}(iy)^{2j}$$
$$\sum_{j=0}^{(n-1)/2} \frac{n!}{(2j+1)!(n-2j-1)!} x^{n-2j-1}(iy)^{2j+1} + \varphi.$$

In both cases the summands are of the form

$$\frac{n!}{k!(n-k)!} x^{n-k}(iy)^k.$$

In the first sum $k = 2j$, where $0 \leq j \leq n/2$, so k is an even integer between 0 and n. In the second $k = 2j+1$, where $0 \leq j \leq (n-1)/2$, so k is an odd integer between 0 and n. Combining, we get a sum over all integers between 0 and n,

$$f(x+iy) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k}(iy)^k + \varphi.$$

By the binomial theorem,

$$f(x+iy) = (x+iy)^n + \varphi$$

or

$$f(z) = z^n + \varphi.$$

Since we are just asked for a single f, we can take $\varphi = 0$ if we want.