MA 2325
Assignment 5
Due 2 December 2009

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1. Let
$$f(z) = \frac{\exp(2\pi i \xi z)}{1+z^2}, \qquad g(z) = \frac{1}{1+z^3},$$
$$h(z) = \frac{\tan(z)}{z^2}, \quad k(z) = \frac{\sin(z)}{z}$$
Find all poles of f, g, h and k.
Solution: In each case we write the function in question as a quotient of functions holomorphic throughout $\mathbf{C}$. All of these are already in this form, except h, which needs to be written as
$$h(z) = \frac{\sin(z)}{z^2\cos(z)}.$$
The zeroes of the denominator are then potentially poles, but may fail to be so if the denominator vanishes as well.

The poles of f are the zeroes of $1+z^2$, namely $\pm i$. The numerator never vanishes, so these are certainly poles. The poles of g are the zeroes of $1+z^3$, which are -1 and $\frac{1\pm i\sqrt{3}}{2}$. Again, the numerator is never zero, so these are all poles. The poles of h are the zeroes of $z^2\cos(z)$, which are 0 and all odd integer multiples of $\frac{\pi}{2}$. The latter are certainly poles, because $\sin(z)$ is non-zero there, but 0 requires further investigation, because the numerator vanishes as well. The first non-zero term in the power series expansion of the numerator is the z term while the first non-zero term in the denominator is the z^2 term, so the quotient has a Laurent expansion starts with a z^{-1} term. We therefore have a pole of order 1 at 0. The function k has *no poles.* The only singularity, at 0, is removable, because the numerator vanishes to the same order as the denominator.

2. With f, g, h and k as above, find the orders of all their poles.
   Solution: In each case we subtract the order of vanishing of the numerator from that of the denominator. As it turns out, we get 1 in each case, so all poles are simple.

3. With f, g, h and k as above, find the residues at each pole.
   Solution: In each case, the simplest way to compute the residue is via the relation
   $$\operatorname*{Res}_{z=w} p(z) = \lim_{z\to w}(z-w)p(z).$$
   This relation holds only for simple poles, but all of the poles in the problem are simple. Simple algebra gives
   $$\operatorname*{Res}_{z=i} f(z) = \frac{\exp(-2\pi\xi)}{2i} \qquad \operatorname*{Res}_{z=-i} f(z) = \frac{\exp(2\pi\xi)}{-2i}$$
   and
   $$\operatorname*{Res}_{z=-1} g(z) = 3$$
   $$\operatorname*{Res}_{z=\frac{1+i\sqrt{3}}{2}} g(z) = 3\frac{-1+i\sqrt{3}}{2} \qquad \operatorname*{Res}_{z=\frac{1-i\sqrt{3}}{2}} g(z) = 3\frac{-1-i\sqrt{3}}{2}$$
   For the pole of h at 0
   $$\operatorname*{Res}_{z=0} h(z) = \lim_{z\to 0}\frac{\tan(z)}{z} = 1.$$
   For the remaining poles of h, we first note that
   $$\begin{aligned}\operatorname*{Res}_{z=\frac{(2n+1)\pi}{2}} \tan(z) &= \lim_{z\to\frac{(2n+1)\pi}{2}}\left(z-\frac{(2n+1)\pi}{2}\right)\tan(z)\\ &= \lim_{\zeta\to 0}\zeta\tan\left(\zeta+\frac{(2n+1)\pi}{2}\right)\\ &= \lim_{\zeta\to 0} -\zeta\cot\zeta = -1.\end{aligned}$$
   Then, by the product formula,
   $$\operatorname*{Res}_{z=\frac{(2n+1)\pi}{2}} z^{-2}\tan(z) = \left(\frac{(2n+1)\pi}{2}\right)^{-2}\operatorname*{Res}_{z=\frac{(2n+1)\pi}{2}}\tan(z) = -\frac{4}{(2n+1)^2\pi^2}$$
   k has no poles, so there is nothing to do.

4. Draw the path

$$\gamma(t) = \frac{2\cos t}{1+\sin^2 t}(1 + i\sin t) \qquad 0 \le t \le 2\pi.$$

What are the winding numbers about the points 1, -1, i, $-i$? You needn't lift any paths or compute any integrals, just use the picture you've drawn.
Solution: The curve is a lemniscate. It looks like a ∞ with ± 1 in the two loops. Looking at the directions, one sees that

$$n(\gamma, 1) = 1 \quad n(\gamma, -1) = -1 \quad n(\gamma, i) = 0 \quad n(\gamma, -i) = 0$$