MA 2325
Assignment 4
Due 18 November 2009

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1. Sine, cosine and tangent are defined by

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \qquad \cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

and

$$\tan(z) = \frac{\sin(z)}{\cos(z)}.$$

Find the power series for $\tan(z)$ up through the z^7 term.
Solution: We have

$$\tan(z)\cos(z) = \sin(z)$$

or

$$\sum_{j=0}^{\infty} a_j z^j \sum_{k=0}^{\infty} b_k z^k = \sum_{l=0}^{\infty} c_l$$

where

$$b_k = (-1)^m/(2m)!$$

if $k = 2m$ and

$$c_l = (-1)^n/(2n+1)!$$

if $l = 2n+1$. The odd b's and even c's are all zero. By the theorem on multiplication of power series,

$$c_l = \sum_{j=0}^{l} a_j b_{l-j}.$$

For $l = 0, 1, \ldots, 7$, we get

$$a_0 = 0$$
$$a_1 = 1$$
$$-\frac{1}{2}a_0 + a_2 = 0$$
$$-\frac{1}{2}a_1 + a_3 = \frac{1}{6}$$
$$\frac{1}{24}a_0 - \frac{1}{2}a_2 + a_4 = 0$$
$$\frac{1}{24}a_1 - \frac{1}{2}a_3 + a_5 = \frac{1}{120}$$
$$-\frac{1}{720}a_0 + \frac{1}{24}a_2 - \frac{1}{2}a_4 + a_6 = 0$$
$$-\frac{1}{720}a_1 + \frac{1}{24}a_3 - \frac{1}{2}a_5 + a_7 = \frac{1}{5040}$$

Solving these equations for the a's,

$$a_1 = 1 \quad a_3 = \frac{1}{3} \quad a_5 = \frac{2}{15} \quad a_7 = \frac{17}{315}.$$

The even a's are all zero. So

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \cdots.$$

2. Find the power series for the function

$$f(z) = (1-z)^{-m}.$$

Hint: Differentiation gives

$$f'(z) = m(1-z)^{-m-1} = m(1-z)^{-1}f(z)$$

or

$$zf'(z) + mf(z) = f'(z).$$

Use the formula for differentiation of power series to determine the coefficients of the power series for f.
Solution: Let a_n be the n' coefficient in the power series expansion of f. From the formula for differentiation of power series, and the equation above, we get

$$\sum_{n=0}^{\infty}(m+n)a_n z^n = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n.$$

The two series must then be equal term by term. To see this, we differentiate n times, set $z = 0$, and then divide by $n!$ to get

$$(m+n)a_n = (n+1)a_{n+1},$$

or

$$a_{n+1} = \frac{m+n}{n+1} a_n.$$

From this we can compute the coefficients successively, starting from

$$a_0 = f(0) = 1.$$

The general formula

$$a_n = \prod_{j=1}^{n} \frac{m+j-1}{j}$$

is easily established by induction.

3. Suppose that f is continuous in an open subset of the complex plane containing the real interval $[a,b]$ and that $\gamma\colon [a,b] \to \mathbf{C}$ is defined by $\gamma(t) = t$. Prove that

$$\int_\gamma f(z)dz = \int_a^b f(x)dx.$$

The integral on the left is a contour integral and the integral on the right is an ordinary real integral.
Hint: This isn't hard, just important.
Solution: By definition,

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b f(t)\,dt.$$

It doesn't matter whether we call the variable of integration t or x.

4. Verify by computing the integral that Cauchy's Theorem holds for the function $f(z) = z^2$ and the triangle

$$T = \{x + iy \in \mathbf{C}\colon x \geq 0, y \geq 0, x + y \leq 1\},$$

i.e. show that

$$\int_{\partial T} f(z)\,dz = 0.$$

Solution: The boundary of T consists of three line segments, from 0 to 1, from 1 to i, and from i to 0. It doesn't matter how we parameterise them, as long as the direction is correct. The simplest choice is

$$\gamma_1 = t \quad \gamma_2 = 1 - t + it \quad \gamma_3(t) = (1-t)i$$

with $0 \leq t \leq 1$ in each case. Then

$$\int_{\gamma_1} z^2\,dz = \int_0^1 t^2\,dt = \frac{1}{3}$$

$$\begin{aligned}
\int_{\gamma_2} z^2\,dz &= \int_0^1 [1-t+it]^2[-1+i]\,dt \\
&= [-1+i]\int_0^1 [1-2t+2it-2it^2]\,dt \\
&= [-1+i][1-1+i-\frac{2}{3}i] = [-1+i]\frac{1}{3}i \\
&= -\frac{1}{3}-\frac{1}{3}i
\end{aligned}$$

$$\begin{aligned}
\int_{\gamma_3} z^2\,dz &= \int_0^1 [(1-t)i]^2[-i]\,dt = -i\int_0^1 [-1+2t-t^2]\,dt \\
&= -i[-1+1-\frac{1}{3}] = \frac{1}{3}i
\end{aligned}$$

The sum of the integrals is then zero, as promised.