MA 2325
Assignment 3
Due 4 November 2009

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1. It was shown in lecture that continuous functions on closed intervals are always uniformly continuous. On open intervals, including infinite intervals, this may not be true.

   (a) Show that $f(x) = x^2$ is *not* uniformly continuous on $\mathbf{R}$.

   (b) Show that $g(x) = \sin(x)$ *is* uniformly continuous on $\mathbf{R}$.

   Solution: To show that f is not uniformly continuous, we need to show that there is a positive ϵ such that for all positive δ there are s and t for which $|s - t| < \delta$ and $|f(s) - f(t)| \geq \epsilon$. It is easy to see that if $\epsilon = 1$, $s = \delta^{-1} + \frac{1}{3}\delta$ and $t = \delta^{-1} - \frac{1}{3}\delta$ then

   $$|s - t| = 2\delta/3 < \delta$$

   and

   $$|f(s) - f(t)| = 4/3 \geq 1.$$

   To show that g is uniformly continuous, we need to show that for all positive ϵ there is a positive δ such that for all s and t, if $|s - t| < \epsilon$ then $|g(s) - g(t)|$. The mean value theorem for derivatives gives a u in between s and t for which

   $$g(s) - g(t) = (s - t)g'(u)$$

   and hence

   $$|g(s) - g(t)| = |s - t||g'(u)|.$$

But $g'(u) = \cos(u)$, so

$$|g'(u)| \leq 1$$

and

$$|g(s) - g(t)| \leq |s - t|.$$

Taking $\delta = \epsilon$, $|s - t| < \delta$ then implies $|g(s) - g(t)| < \epsilon$.

2. Compute, from the definition, the winding number of the path

$$\gamma_{n,w}(t) = \exp(2\pi i n t) + w \quad 0 \leq t \leq 1$$

about w, where n is an integer.
Solution: We need to find a path $\tilde{\gamma}_{n,w}$ such that

$$\exp(\tilde{\gamma}_{n,w}(t)) = \gamma_{n,w}(t) - w$$

for all $t \in [0, 1]$, in other words, we need

$$\exp(\tilde{\gamma}_{n,w}(t)) = \exp(2\pi i n t).$$

The obvious choice is

$$\tilde{\gamma}_{n,w}(t) = 2\pi i n t.$$

The winding number is then

$$\frac{\tilde{\gamma}_{n,w}(1) - \tilde{\gamma}_{n,w}(0)}{2\pi i} = n.$$

3. Compute the contour integral

$$\int_{\gamma_{n,w}} \frac{dz}{z - w}$$

from the definition, where the path $\gamma_{n,w}$ is the path defined in the preceding problem.
Solution: By definition

$$\int_{\gamma_{n,w}} \frac{dz}{z - w} = \int_0^1 \frac{\gamma'_{n,w}(t)}{\gamma_{n,w}(t) - w} \, dt = \int_0^1 2\pi i n \, dt = 2\pi i n.$$

4. Show that for any closed path γ and point w not on γ,

$$\int_\gamma \frac{dz}{z - w} = 2\pi i n(\gamma, w).$$

Solution: We choose, using the Path Lifting Theorem, a path $\tilde{\gamma}_w$ such that

$$\exp(\tilde{\gamma}_w(t)) = \gamma(t) - w$$

for all $t \in [a, b]$. Differentiating using the chain rule,

$$\exp'(\tilde{\gamma}_w(t))\tilde{\gamma}'_w(t) = \gamma'(t).$$

or, since $\exp' = \exp$,

$$\exp(\tilde{\gamma}_w(t))\tilde{\gamma}'_w(t) = \gamma'(t).$$

from which

$$\gamma'(t) = (\gamma(t) - w)\tilde{\gamma}'_w(t).$$

By definition,

$$\int_\gamma \frac{dz}{z - w} = \int_a^b \frac{\gamma'(t)}{\gamma(t) - w}\, dt.$$

Substituting, using the Fundamental Theorem of the Calculus, and then the definition of the winding number,

$$\int_\gamma \frac{dz}{z - w} = \int_a^b \tilde{\gamma}'(t)\, dt = \tilde{\gamma}(b) - \tilde{\gamma}(a) = 2\pi i n(\gamma, w).$$