MA 2325
Assignment 2
Due 21 October 2009

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1. In lecture it was proved that if both $\sum_{j=0}^{\infty} a_j$ and $\sum_{k=0}^{\infty} b_k$ are absolutely convergent then $\sum_{l=0}^{\infty} c_l$ is convergent where $c_l = \sum_{j=0}^{l} a_j b_{l-j}$, and that

$$\sum_{l=0}^{\infty} c_l = \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right).$$

Prove that $\sum_{l=0}^{\infty} c_l$ converges absolutely.
Hint: The most straightforward way to prove this is to use the theorem that every bounded increasing sequence of real numbers is convergent.
Solution: Saying that $\sum_{l=0}^{\infty} c_l$ converges absolutely is the same as saying that $\sum_{l=0}^{\infty} |c_l|$ converges. This is a series with non-negative summands, so the partial sums are an increasing sequence. Using the theorem referred to in the hint, all we need to prove is that the partial sums are bounded. But

$$\begin{aligned}\sum_{l=0}^{n} |c_l| &= \sum_{l=0}^{n} \left|\sum_{j=0}^{l} a_j b_{l-j}\right| \\ &\le \sum_{l=0}^{n} \sum_{j=0}^{l} |a_j||b_{l-j}| \\ &= \sum_{j+k\le n} |a_j||b_k|\end{aligned}$$

by the triangle inequality. If j and k satisfy $j+k \le n$ then $j \le n$ and $k \le n$. Adding further non-negative terms can only increase the sum, so

$$\sum_{l=0}^{n} |c_l| \le \sum_{j=0}^{n} |a_j| \sum_{k=0}^{n} |b_k|.$$

Since $\sum_{j=0}^{\infty} a_j$ and $\sum_{k=0}^{\infty} b_k$ are absolutely convergent, the partial sums for $\sum_{j=0}^{\infty} |a_j|$ and $\sum_{k=0}^{\infty} |b_k|$ are convergent and hence bounded. In other words, there are A and B such that

$$\sum_{j=0}^{n} |a_j| \leq A$$

and

$$\sum_{k=0}^{n} |b_k| \leq B$$

for all n. Then

$$\sum_{l=0}^{n} |c_l| \leq AB$$

for all n, so we are done.

2. There is a power series $\sum_{k=0}^{\infty} b_k z^k$ such that

$$(\exp(z) - 1) \sum_{n=0}^{\infty} b_k z^k = z.$$

Find b_k for $k = 0, 1, \ldots, 7$.
Solution: We have an equation of the form

$$\sum_{j=0}^{\infty} a_j (z-w)^j \sum_{k=0}^{\infty} b_k (z-w)^k = \sum_{l=0}^{\infty} c_l (z-w)^l$$

where $w = 0$,

$$a_j = \begin{cases} 0 & \text{if } j = 0, \\ 1/j! & \text{otherwise,} \end{cases}$$

and

$$c_l = \begin{cases} 1 & \text{if } l = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By the theorem on multiplication of power series,

$$c_l = \sum_{j=0}^{l} a_j b_{l-j}$$

for all l. For $l = 0$ this gives only the useless equation $0 = 0$. For $l = 1$ we get

$$b_0 = 1.$$

For $l > 1$ we get

$$0 = \sum_{j=0}^{l} a_j b_{l-j} = b_{l-1} + \sum_{j=2}^{l} b_{l-j}/j!.$$

and hence

$$b_{l-1} = -\sum_{j=2}^{l} b_{l-j}/j!.$$

Applying this equation for $j = 2, 3 \dots 8$ gives

$$b_1 = -1/2, \quad b_2 = 1/12, \quad b_4 = -1/720, \quad b_6 = 1/30240.$$

The remaining coefficients, b_3, b_5 and b_7, are all zero.

3. Prove that for real y,

$$\exp(iy) = \cos(y) + i\sin(y)$$

and that for real x and y,

$$\exp(x + iy) = \exp(x)\cos(y) + i\exp(x)\sin(y).$$

Solution: The second equation follows immediately from the first and the addition formula for the exponential function. To prove the first equation we start from the definitions of the exponential and of infinite series.

$$\exp(iy) = \sum_{j=0}^{\infty} \frac{(iy)^j}{j!} = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{(iy)^j}{j!}$$

Now $(iy)^n = i^n y^n$ and an easy induction shows that

$$i^n = \begin{cases} (-1)^k & \text{if } n = 2k, \\ (-1)^k i & \text{if } n = 2k+1. \end{cases}$$

Thus

$$\exp(iy) = \lim_{n\to\infty} \left(\sum_{2k\le n} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{2k+1\le n} (-1)^k \frac{y^{2k+1}}{(2k+1)!} \right).$$

We know that if $\zeta_n = \xi_n + i\eta_n$, with ξ_n and η_n real then ζ converges if and only if ξ and η do, in which case

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \xi_n + i \lim_{n\to\infty} \eta_n.$$

Applying this to

$$\xi_n = \sum_{2k\le n} (-1)^k \frac{y^{2k}}{(2k)!}, \quad \eta_n = \sum_{2k+1\le n} (-1)^k \frac{y^{2k+1}}{(2k+1)!}, \quad \zeta_n = \xi_n + i\eta_n$$

gives

$$\exp(iy) = \sum_{k=0}^{\infty}(-1)^k \frac{y^{2k}}{(2k)!} + i\sum_{k=0}^{\infty}(-1)^k \frac{y^{2k+1}}{(2k+1)!} = \cos(y) + i\sin(y).$$

This proof is a bit long, but it avoids any worries about rearranging the order of summation. Such a rearrangement is justified in this case because the exponential series converges *absolutely*.

4. Prove the algebraic identity used in lecture

$$\begin{aligned}(s-w)^n - (z-w)^n &= n(s-z)(z-w)^{n-1} \\ &\quad + (s-z)^2 \sum_{k=0}^{n-2}(n-k-1)(s-w)^k(z-w)^{n-k-2}.\end{aligned}$$

Hint: Start with the special case $w = 0$. Use induction, possibly more than once.
Solution: As the hint suggests, we start from the special case $w = 0$, that is

$$s^n - z^n = n(s-z)z^{n-1} + (s-z)^2 \sum_{k=0}^{n-2}(n-k-1)s^k z^{n-k-2}.$$

There are several ways to prove this, some easier than others, but all use induction at some stage. The easiest is probably to start with the simpler equation

$$s^n - z^n = (s-z)\sum_{j=0}^{n-1} s^j z^{n-j-1}.$$

This is certainly true for $n = 0$, in which case the sum on the left is empty. If it's true for some n then

$$\begin{aligned}s^{n+1} - z^{n+1} &= s^n(s-z) + (s^n - z^n)z \\ &= (s-z)s^n + (s-z)\left(\sum_{j=0}^{n-1} s^j z^{n-j-1}\right) z \\ &= (s-z)s^n + (s-z)\left(\sum_{j=0}^{n-1} s^j z^{n-j}\right) \\ &= (s-z)\left(s^n + \sum_{j=0}^{n-1} s^j z^{n-j}\right) \\ &= (s-z)\sum_{j=0}^{n} s^j z^{n-j}.\end{aligned}$$

Thus

$$s^{n+1} - z^{n+1} = (s-z)\sum_{j=0}^{n} s^j z^{n-j},$$

the equation we started from, with $n+1$ in place of n. The equation is thus true by induction for all n. It's still not what we wanted to prove, however. Next, $s^j = (s^j - z^j) + z^j$, so

$$\begin{aligned} s^n - z^n &= (s-z)\sum_{j=0}^{n-1} s^j z^{n-j-1} \\ &= (s-z)\sum_{j=0}^{n-1}(s^j - z^j)z^{n-j-1} + (s-z)\sum_{j=0}^{n-1} z^j z^{n-j-1}. \end{aligned}$$

The second sum on the right is has n terms, each equal to z^{n-1}, so

$$s^n - z^n = n(s-z)z^{n-1} + (s-z)\sum_{j=0}^{n-1}(s^j - z^j)z^{n-j-1}.$$

We already know that

$$s^j - z^j = (s-z)\sum_{k=0}^{j-1} s^k z^{j-k-1},$$

so

$$s^n - z^n = n(s-z)z^{n-1} + (s-z)^2 \sum_{j=0}^{n-1}\sum_{k=0}^{j-1} s^k z^{n-k-2}.$$

We then reverse the order of summation. The possible values of j and k are given by the inequalities $0 \le k < j < n$, so k ranges from 0 to $n-2$ and j from $k+1$ to $n-1$,

$$s^n - z^n = n(s-z)z^{n-1} + (s-z)^2 \sum_{k=0}^{n-2}\sum_{j=k+1}^{n-1} s^k z^{n-k-2}.$$

The inner sum has $n-k-1$ identical terms, so we are left with

$$s^n - z^n = n(s-z)z^{n-1} + (s-z)^2 \sum_{k=0}^{n-2}(n-k-1)s^k z^{n-k-2}.$$

This holds for all complex s and z. We may therefore substitute for s and z any complex expressions. In particular, we may substitute $s-w$ and $z-w$. Doing so gives the equation we were seeking to prove.