MA 2325
Assignment 1
Due 7 October 2009

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1. Defining the complex conjugate in the usual way, $\overline{x+iy} = x - iy$, prove that
$$\lim_{n\to\infty} a_n = L$$
if and only if
$$\lim_{n\to\infty} \overline{a_n} = \overline{L}.$$
Solution:

First, a work about *non*-proofs. The usual algebraic properties of limits would imply
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$
where $a_n = x_n + iy_n$. That is, they *would* imply it *if* we knew that the limits
$$\lim_{n\to\infty} x_n$$
and
$$\lim_{n\to\infty} y_n$$
existed. There was a theorem to this effect, but it's proof depended on the result of *this* question, so to use it would be circular reasoning.

Now for a real proof. First the "if" part.
$$\lim_{n\to\infty} \overline{a_n} = \overline{L}$$
means that for all $\epsilon > 0$ there is an N such that for all $n > N$
$$|\overline{a}_n - \overline{L}| < \epsilon.$$

But

$$|\overline{a}_n - \overline{L}| = |a_n - L|$$

so

$$|a_n - L| < \epsilon$$

for all $n > N$. Thus

$$\lim_{n\to\infty} a_n = L.$$

The proof of the "only if" part is nearly identical, but with the equations in reverse order.

2. Recall that an open set in the complex plane is, by definition, one which contains a disc of positive radius about each of its points. Prove that

   (a) the intersection of two open sets is an open set.
   Solution: Suppose U_1 and U_2 are open sets. If $w \in U_1 \cap U_2$ then, since U_1 is open, there is an $r_1 > 0$ such that $D_{r_1,w} \subset U_1$. Similarly there is an $r_2 > 0$ such that $D_{r_2,w} \subset U_2$. Then $D_{r,w} \subset U_1 \cap U_2$, where $r = \min(r_1, r_2) > 0$. Since $w \in U_1 \cap U_2$ was arbitrary we see that $U_1 \cap U_2$ contains a disc of positive radius about each of its points, and hence is open.

   Note that there's no need to consider the case $U_1 \cap U_2 =$ separately. It's not wrong to do so, but it's pointless.

   (b) the union of arbitrarily many open sets is an open set.
   Solution:

   First note that "arbitrarily many" really means arbitrarily many. You should *not* assume that the number is finite. Any argument based on induction on the number of open sets is therefore doomed.

   Suppose S is a collection of open sets and $w \in \cup_{V\in S} V$. By the definition of the union, $w \in W$ for some particular $W \in S$. Since $W \in S$, W is open and hence there is an $r > 0$ such that $D_{r,w} \subset W$. On the other hand, from $W \in S$ it follows that $W \subset \cup_{V\in S} V$. Then $D_{r,w} \subset \cup_{V\in S} V$. Since $w \in \cup_{V\in S} V$ was arbitrary we see that the set $\cup_{V\in S} V$ contains a disc of positive radius about each of its points, *i.e.* that it is open.

3. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} n^{-1} z^n.$$

Solution: The radius of convergence is 1. The easiest way to prove this is to note that the series converges when $z = -1$, by the alternating series test, and diverges for $z = 1$, since it is then the harmonic series. By the first version of the Comparison Lemma from Lecture 3 the series converges for $|z| < |-1| = 1$. By the second version of the same lemma, it diverges for $|z| > |1| = 1$.

It's also possible to prove this by the ratio test. I avoided doing so above because we haven't done the ratio test for complex series, but I haven't marked such arguments wrong.