1 Lagrangian Mechanics

The usual formulation of Lagrangian Mechanics distinguishes one coordinate, time, and considers all spatial variables as functions of time. This function is constrained to be stationary for the action

$$\int L(t, x^\alpha, \dot{x}^\alpha)\, dt,$$

where $t = x^0$ is the distinguished coordinate, x^1, …, x^n are the other coordinates[1], dots are used to denote t derivatives, and L is the Lagrangian.

From a relativistic viewpoint it is more natural to consider the trajectory as a curve in spacetime, and allow parameterisations other than by the coordinate t. For example, many formulae become simpler when arc length is used as a parameter. In this viewpoint, t and x^α are functions of some parameter s,

$$\dot{x}^\alpha = \frac{dx^\alpha}{dt} = \frac{dx^\alpha}{ds}\Big/\frac{dt}{ds} = \frac{v^\alpha}{v^0},$$

where $v^\alpha = dx^\alpha/ds$ and $v^0 = dx^0/ds = dt/ds$. The action is then, by a simple change of variable,

$$\int L(t, x^\alpha, v^\alpha/v^0)v^0\, ds = \int \mathcal{L}(t, x^\alpha, v^0, v^\alpha)\, ds,$$

where

$$\mathcal{L}(t, x^\alpha, v^0, v^\alpha) = L(t, x^\alpha, v^\alpha/v^0)v^0.$$

It is $\mathcal{L}$ rather than L which behaves nicely under coordinate changes which mix time and space variables.

The Euler-Lagrange equations for this action are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha}$$

in the classical picture, or

$$\frac{d}{ds}\frac{\partial \mathcal{L}}{\partial v^a} = \frac{\partial \mathcal{L}}{\partial x^a}$$

in the spacetime picture. Note that the index is a Latin letter in the second set of equation, so the index 0 is allowed. There is an important difference

---

[1] Greek letters are used for indices other than 0. Where such an index appears as both an upper and lower index, the Einstein summation convention applies, but the index 0 excluded from the sum.

between these two sets of equations. We expect the first set to have unique solutions for physically reasonable problems. The second set should *not* have unique solutions, since it is only the curve in spacetime and not the parameterisation which has physical meaning. To get unique solutions we must impose some additional condition to fix the parameterisation.

2 Hamiltonian Mechanics

One of the main points of any Classical Mechanics course is to show that the Lagrangian system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha}$$

is equivalent to Hamilton's Canonical Equations

$$\frac{dx^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha} \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x^\alpha},$$

where the canonical momenta p_α are defined by

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha}$$

and the Hamiltonian is defined by

$$H = p_\alpha \dot{x}^\alpha - L.$$

In computing partial derivatives, H is to be understood as a function of t, x^α and p_α rather than t, x^α and $\dot{x}^\alpha$. I won't repeat the proof here, but the standard argument assumes almost nothing about the form of L. In particular, L need not be the difference between kinetic and potential energy. The only thing we need is strict convexity in the variables $\dot{x}^\alpha$, to ensure that we can solve for the velocities in terms of the momenta.

Computing the partial derivatives $\mathcal{L}$ using the chain rule,

$$\frac{\partial \mathcal{L}}{\partial v^0} = L(t, x^\alpha, v^\alpha/v^0) - \frac{\partial L}{\partial \dot{x}^\alpha}\frac{v^\alpha}{v^0}$$

and

$$\frac{\partial \mathcal{L}}{\partial v^\alpha} = \frac{\partial L}{\partial \dot{x}^\alpha}.$$

We can write both these equations as

$$p_a = \frac{\partial \mathcal{L}}{\partial v^a}$$

if we adopt the convention that

$$p_0 = -H.$$

With this convention,

$$\frac{dp_a}{dt} = -\frac{\partial H}{\partial x^a}.$$

The new equation, the one for $a = 0$, is however a consequence of the canonical equations. It is the classical relation

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

3 Symmetry

We can think of θ and φ as coordinates on a sphere. Suppose now that our Hamiltonian is invariant under rotations of that sphere. That happens if and only if it is of the form

$$H(t, r, \theta, \varphi, p_r, p_\theta, p_\varphi) = h\left(t, r, p_r, \lambda\right),$$

where

$$\lambda = \frac{p_\theta^2 + \csc^2\theta p_\varphi^2}{r^2},$$

for some function h, where t and r are two other coordinates. There are various ways we can use this fact to simplify things.

The approach I will take here is to replace the angular part of our original coordinate system with new coordinates $\tilde{\varphi}$ and $\tilde{\theta}$, such that $\tilde{\theta} = \frac{\pi}{2}$ for all time. It's possible to write this coordinate transformation in coordinates, but very unenlightening. The idea, however, is simple. The initial position and velocity belong to a plane, and the subsequent motion must be confined to that plane. If it were not then reflecting through that plane would give distinct solutions to the initial value problem, violating the theorem on existence and uniqueness of solutions of systems of ordinary differential equations. By rotating the coordinate sphere we can make this plane to be the plane $\tilde{\theta} = \frac{\pi}{4}$. We then forget about the original coordinate system and drop the ˜'s, since we will not need to go back to our orginal angular coordinates.

4 Relativistic Kepler

In general, for a particle of mass m and charge q in a spacetime with metric g and electromagnetic field Φ, the relativistic Lagrangian is[2]

$$-\mathcal{L} = mcw + q\Phi_a v^a,$$

where

$$w = \sqrt{-g_{ab}v^a v^b}.$$

The momentum covector is then

$$p_a = mcg_{ab}v^b/w - q\Phi_a$$

and satisfies

$$g^{ab}(p_a + q\Phi_a)(p_b + q\Phi_b) = -m^2c^2$$

With

$$g_{tt} = -c^2\Delta, \quad g_{tt} = \Delta^{-1}, \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = r^2\sin^2\theta, \quad \Phi_t = -\frac{Q}{r},$$

$$\Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

the Hamiltonian is

$$H = \frac{mc^2\Delta}{\sqrt{\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{r^2\dot{\theta}^2}{c^2} - \frac{r^2\sin^2\theta\dot{\varphi}^2}{c^2}}} + \frac{Qq}{r}.$$

The canonical momenta are

$$p_r = \frac{m\Delta^{-1}\dot{r}}{\sqrt{\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{r^2\dot{\theta}^2}{c^2} - \frac{r^2\sin^2\theta\dot{\varphi}^2}{c^2}}}$$

$$p_\theta = \frac{mr^2\dot{\theta}}{\sqrt{\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{r^2\dot{\theta}^2}{c^2} - \frac{r^2\sin^2\theta\dot{\varphi}^2}{c^2}}}.$$

and

$$p_\varphi = \frac{mr^2\sin^2\theta\dot{\varphi}}{\sqrt{\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{r^2\dot{\theta}^2}{c^2} - \frac{r^2\sin^2\theta\dot{\varphi}^2}{c^2}}}.$$

---

[2] The choice of sign makes no difference in the equations of motion. The choice here makes the momentum tend to its classical value in the limit $c \to \infty$.

These satisfy the relation

$$c^{-2}\Delta^{-1}\left(H - \frac{Qq}{r}\right)^2 - \Delta p_r^2 - r^{-2}p_\theta^2 - r^{-2}\csc^2\theta p_\varphi^2 = m^2c^2.$$

We may, however, assume without loss of generality that $\theta = \frac{\pi}{2}$, so the equations above reduce to

$$H = \frac{mc^2\Delta}{\sqrt{\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{r^2\dot{\varphi}^2}{c^2}}} + \frac{Qq}{r}.$$

$$p_r = \frac{m\Delta^{-1}\dot{r}}{\sqrt{\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{r^2\dot{\varphi}^2}{c^2}}c^2}$$

$$p_\varphi = \frac{mr^2\sin^2\theta\dot{\varphi}}{\sqrt{\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{r^2\dot{\varphi}^2}{c^2}}}$$

and

$$c^{-2}\Delta^{-1}\left(H - \frac{Qq}{r}\right)^2 - \Delta p_r^2 - r^{-2}p_\varphi^2 = m^2c^2.$$

These are the equations we would get from the metric

$$g_{tt} = -c^2\Delta, \quad g_{tt} = \Delta^{-1}, \quad g_{\varphi\varphi} = r^2\theta$$

in a three dimensional spacetime with coordinates t, r and φ.

5 Circular Orbits

The circular orbits are the ones for which

$$\dot{r} = \partial H/\partial p_r = 0$$

and

$$\dot{p}_r = -\partial H/\partial r = 0.$$

Differentiating

$$c^{-2}\Delta^{-1}\left(H - \frac{Qq}{r}\right)^2 - \Delta p_r^2 - r^{-2}p_\varphi^2 = m^2c^2.$$

These give the equations

$$p_r = 0$$

and

$$-c^{-2}\Delta^{-2}\Delta_r\left(H - \frac{Qq}{r}\right)^2 + 2c^{-2}\Delta^{-1}\left(H - \frac{Qq}{r}\right)\frac{Qq}{r^2} + 2r^{-3}p_\varphi^2 = 0.$$

We can solve the three equations above to get H and p_θ as functions of r. This allows us to determine the period of rotation as a function of r. This is somewhat complicated in general, so I will do only the special case $Q = q = 0$. Eliminating H first,

$$p_\varphi^2 = -m^2r^2c^2\frac{r\Delta_r}{r\Delta_r - 2\Delta} = m^2r^2c^2\frac{M}{r - 3M}$$

Then

$$H^2 = c^2\Delta\left(m^2c^2 + r^{-2}p_\varphi^2\right) = m^2c^4\Delta\frac{r - 2M}{r - 3M}$$

$$\dot{\varphi}^2 = \left(\frac{c^2\Delta}{r^2}\frac{p_\varphi}{H}\right)^2 = \frac{Mc^2}{r^3}$$

so the orbital period is

$$\frac{2\pi}{\dot{\varphi}} = \frac{2\pi}{\sqrt{\frac{Mc^2}{r^3}}}$$

Classically, the orbital period is

$$\frac{2\pi}{\sqrt{G\overline{m}}r^3}$$

where $\overline{m}$ is the central mass and G is the Newtonian gravitational constant. To ensure that we get the correct behaviour in the classical limit we therefore require

$$M = \frac{G\overline{m}}{c^2}$$

If we do the more complicated calculation with Q and q non-zero then we get

$$Q^2 = \frac{G\overline{q}^2}{4\pi\epsilon_0c^4},$$

as well, where $\overline{q}$ is the central charge and ϵ_0 is the permittivity of the vacuum.

6 Angle of Deflection

The classical experimental tests of General Relativity are based on angles of deflection caused by the gravitational field of the sun, either for massless particles (light in the case of gravitational lensing in eclipses) or massive (the perihelion shift of Mercury). The mathematical apparatus for the two cases is identical. Again we take $Q = q = 0$.

$$\frac{p_\varphi}{H} = \frac{r^2\dot{\varphi}}{c^2\Delta}$$

and hence

$$\dot{\varphi} = \frac{c^2\Delta}{r^2}\frac{p_\varphi}{H}$$

and

$$\frac{r^2\dot{\varphi}^2}{c^2} = \frac{c^2\Delta^2}{r^2}\frac{p_\varphi^2}{H^2}.$$

Then

$$H^2 = \frac{m^2c^4\Delta^2}{\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{c^2\Delta^2}{r^2}\frac{p_\varphi^2}{H^2}},$$

$$\Delta - \frac{\dot{r}^2}{c^2\Delta} - \frac{c^2\Delta^2}{r^2}\frac{p_\varphi^2}{H^2} = \frac{m^2c^4\Delta^2}{H^2},$$

and

$$\dot{r}^2 = c^2\Delta\left(\Delta - \frac{c^2\Delta^2}{r^2}\frac{p_\varphi^2}{H^2} - \frac{m^2c^4\Delta^2}{H^2}\right).$$

Therefore

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{\dot{r}^2}{\dot{\varphi}^2} = r^4\left(\frac{H^2 - m^2c^4\Delta}{p_\varphi^2c^2} - \frac{\Delta}{r^2}\right).$$

Making the change of variable $u = 1/r$,

$$\Delta = 1 - 2Mu,$$

$$\frac{du}{d\varphi} = -\frac{1}{r^2}\frac{dr}{d\varphi}$$

and hence

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{H^2 - m^2c^4\Delta}{p_\varphi^2c^2} - \Delta u^2$$

or

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{H^2 - m^2c^4}{p_\varphi^2 c^2} + 2M\frac{m^2c^2}{p_\varphi^2}u - u^2 + 2Mu^3.$$

Then the change in angle between different values of u is

$$\int d\varphi = \int \frac{du}{\sqrt{\frac{H^2-m^2c^4}{p_\varphi^2 c^2} + 2M\frac{m^2c^2}{p_\varphi^2}u - u^2 + 2Mu^3}}.$$

There are two cases of primary interest. For scattering,

$$H^2 > m^2c^4$$

and it makes sense to integrate from $u = 0$ to the first zero of the cubic

$$\frac{H^2 - m^2c^4}{p_\varphi^2 c^2} + 2M\frac{m^2c^2}{p_\varphi^2}u - u^2 + 2Mu^3$$

to obtain the change in deflection of a particle at it comes in from infinity to perihelion. Doubling this angle gives the deflection from infinity to infinity. In theory this applies to any particle, but it is only observable for light, with $m = 0$. One advantage of the Hamiltonian over the Lagrangian approach is that it copes better with the case $m = 0$. The other interesting case is

$$H^2 < m^2c^4.$$

Then $u = 0$, *i.e.* $r = \infty$ is not allowed and we are dealing with a bound orbit. This clearly requires $m \neq 0$. In this case we integrate between successive zeroes of

$$\frac{H^2 - m^2c^4}{p_\varphi^2 c^2} + 2M\frac{m^2c^2}{p_\varphi^2}u - u^2 + 2Mu^3$$

to get the angle between perihelion and aphelion, or double the integral to get the angle between successive perihelions. Classically that angle is 2π. Relativistically there is a small but measurable shift.