MA 4448
Assignment 4
Due 31 March 2011

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1. The Kerr metric is, in coordinates t, r, θ, φ,

$$g_{tt} = -c^2 \left(1 - \frac{2G\overline{m}r}{c^2 r^2 \left(1 + \frac{j^2 \cos^2\theta}{\overline{m}^2 c^2 r^2} \right)} \right)$$

$$g_{t\varphi} = g_{\varphi t} = -\frac{2Gj \sin^2\theta}{c^2 r \left(1 + \frac{j^2 \cos^2\theta}{\overline{m}^2 c^2 r^2} \right)}$$

$$g_{rr} = \frac{1 + \frac{j^2 \cos^2\theta}{c^2 \overline{m}^2 r^2}}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2 \overline{m}^2 r^2}}$$

$$g_{\theta\theta} = r^2 \left(1 + \frac{j^2 \cos^2\theta}{\overline{m}^2 c^2 r^2} \right)$$

$$g_{\varphi\varphi} = \left(r^2 + \frac{j^2}{c^2 \overline{m}^2} + \frac{2Gj^2 \sin^2\theta}{c^4 \overline{m} r \left(1 + \frac{j^2 \cos^2\theta}{\overline{m}^2 c^2 r^2} \right)} \right) \sin^2\theta.$$

Here J is the angular momentum of the central mass and $\overline{m}$ is its mass. G is the Newtonian gravitational constant. All other entries of the metric tensor are zero. Find the redshift between an arbitrary pair of stationary observers.
Note: Don't be frightened by the hideousness of the metric. The problem is in fact really easy.
Solution: Suppose the sender emits a signal starting the event $(t_s, r_s, \theta_s, \varphi_s)$

which is received by an observer at the event $(t_o, r_o, \theta_o, \varphi_o)$. If the frequency as seen by the sender is ν_s then the signal repeats after a proper time of $1/\nu_s$. That is at the event $(t_s + \tau, r_s, \theta_s, \varphi_s)$, where

$$g_{tt}(t_s, \theta_s)\tau^2 = -c^2/\nu_s^2.$$

The other coordinates are the same, because we have assumed the sender and receiver are stationary. The receiver sees one period elapse at the event $(t_o + \tau, r_o, \theta_o, \varphi_o)$. The t-interval must be the same, since translation in the t variable is a symmetry, so the path followed by the later signal is just the translate of the path followed by the earlier signal. As seen by the receiver, the frequency is given by

$$g_{tt}(t_o, \theta_o)\tau^2 = -c^2/\nu_o^2.$$

The frequencies are related by

$$\frac{\nu_o}{\nu_s} = \sqrt{\frac{g_{tt}(t_s, \theta_s)}{g_{tt}(t_o, \theta_o)}} = \sqrt{\frac{1 - \frac{2G\overline{m}r_s}{c^2 r_s^2\left(1 + \frac{j^2\cos^2\theta_s}{\overline{m}^2 c^2 r_s^2}\right)}}{1 - \frac{2G\overline{m}r_o}{c^2 r_o^2\left(1 + \frac{j^2\cos^2\theta_o}{\overline{m}^2 c^2 r_o^2}\right)}}}$$

If we assume the signals propagate at the speed of light then we can solve for the actual trajectories, but that is quite complicated and completely unnecessary.

2. For the Kerr metric, as for the Reissner-Nordström, $\theta \to \pi - \theta$ is a symmetry, so if $\theta = \pi/2$ initially and $\dot{\theta} = 0$ then $\theta = \pi/2$ for all time. We *don't* have full rotational symmetry so we can *not* say that $\theta = \pi/2$ without loss of generality. But the case $\theta = \pi/2$ is still physically interesting. This problem is equivalent to studying the motion of a particle in a three dimensional space time with coordinates t, r and φ with metric tensor obtained by setting $\theta = \pi/2$ in the equations above and then dropping $g_{\theta\theta}$:

$$g_{tt} = -c^2\left(1 - \frac{2G\overline{m}}{c^2 r}\right)$$

$$g_{t\varphi} = g_{\varphi t} = -\frac{2Gj}{c^2 r}$$

$$g_{rr} = \frac{1}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}}$$

$$g_{\varphi\varphi} = \left(r^2 + \frac{j^2}{c^2\overline{m}^2} + \frac{2Gj^2}{c^4\overline{m}r} \right).$$

What is the inverse metric?
Solution: This is just an ugly matrix inversion. The metric has matrix

$$\begin{pmatrix} -c^2\left(1 - \frac{2G\overline{m}}{c^2r}\right) & 0 & -\frac{2Gj}{c^2r} \\ 0 & \frac{1}{1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2}} & 0 \\ -\frac{2Gj}{c^2r} & 0 & r^2 + \frac{j^2}{c^2\overline{m}^2} + \frac{2Gj^2}{c^4\overline{m}r} \end{pmatrix}$$

The inverse matrix is

$$\begin{pmatrix} -c^{-2}\frac{1+\frac{j^2}{c^2\overline{m}^2r}+\frac{2Gj^2}{c^4\overline{m}r^3}}{1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2}} & 0 & -\frac{2Gj}{c^4r^3\left(1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2}\right)} \\ 0 & 1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2} & 0 \\ -\frac{2Gj}{c^4r^3\left(1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2}\right)} & 0 & \frac{1}{r^2}\frac{1-\frac{2G\overline{m}}{c^2r}}{1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2}} \end{pmatrix}$$

So the inverse metric is

$$g^{tt} = -c^{-2}\frac{1+\frac{j^2}{c^2\overline{m}^2r}+\frac{2Gj^2}{c^4\overline{m}r^3}}{1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2}}$$

$$g^{t\varphi} = -\frac{2Gj}{c^4r^3\left(1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2}\right)}$$

$$g^{rr} = 1 - \frac{2G\overline{m}}{c^2r} + \frac{j^2}{c^2\overline{m}^2r^2}$$

$$g^{\varphi\varphi} = \frac{1}{r^2}\frac{1-\frac{2G\overline{m}}{c^2r}}{1-\frac{2G\overline{m}}{c^2r}+\frac{j^2}{c^2\overline{m}^2r^2}}$$

3. What are the canonical momenta and Hamiltonian for an uncharged test particle of mass m on a spacetime with metric as in the previous problem? What algebraic relation do they satisfy?
Solution:

Using the general formulae

$$w = \sqrt{-g_{ab}v^av^b},$$

$$p_a = mcg_{ab}v^b/w$$

and

$$\dot{x}^\alpha = v^\alpha/v^0,$$

$$
\begin{aligned}
H = -p_t &= -mc\frac{g_{tt} + g_{t\varphi}\dot{\varphi}}{\sqrt{-g_{tt} - 2g_{t\varphi}\dot{\varphi} - g_{rr}\dot{r}^2 - g_{\varphi\varphi}\dot{\varphi}^2}} \\
&= \frac{mc^2\left(1 - \frac{2G\overline{m}}{c^2 r}\right) + m\frac{2Gj}{c^2 r}\dot{\varphi}}{\sqrt{\left(1 - \frac{2G\overline{m}}{c^2 r}\right) + \frac{4Gj}{c^2 r}\frac{\dot{\varphi}}{c^2} - \frac{1}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}}\frac{\dot{r}^2}{c^2} - r^2\left(1 + \frac{j^2}{c^2\overline{m}^2 r^2} + 2\frac{Gj^2}{c^4\overline{m}^2 r^3}\right)\frac{\dot{\varphi}^2}{c^2}}},
\end{aligned}
$$

$$
\begin{aligned}
p_r &= mc\frac{g_r r\dot{r}}{\sqrt{-g_{tt} - 2g_{t\varphi}\dot{\varphi} - g_{rr}\dot{r}^2 - g_{\varphi\varphi}\dot{\varphi}^2}} \\
&= \frac{\frac{m\dot{r}}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}}}{\sqrt{\left(1 - \frac{2G\overline{m}}{c^2 r}\right) + \frac{4Gj}{c^2 r}\frac{\dot{\varphi}}{c^2} - \frac{1}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}}\frac{\dot{r}^2}{c^2} - r^2\left(1 + \frac{j^2}{c^2\overline{m}^2 r^2} + 2\frac{Gj^2}{c^4\overline{m}^2 r^3}\right)\frac{\dot{\varphi}^2}{c^2}}}
\end{aligned}
$$

and

$$
\begin{aligned}
p_\varphi &= mc\frac{g_{t\varphi} + g_{\varphi\varphi}\dot{\varphi}}{\sqrt{-g_{tt} - 2g_{t\varphi}\dot{\varphi} - g_{rr}\dot{r}^2 - g_{\varphi\varphi}\dot{\varphi}^2}} \\
&= \frac{mr^2\left(1 + \frac{j^2}{c^2\overline{m}^2 r^2} + 2\frac{Gj^2}{c^4\overline{m}^2 r^3}\right)\dot{\varphi} - \frac{2Gmj}{c^2 r}}{\sqrt{\left(1 - \frac{2G\overline{m}}{c^2 r}\right) + \frac{4Gj}{c^2 r}\frac{\dot{\varphi}}{c^2} - \frac{1}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}}\frac{\dot{r}^2}{c^2} - r^2\left(1 + \frac{j^2}{c^2\overline{m}^2 r^2} + 2\frac{Gj^2}{c^4\overline{m}^2 r^3}\right)\frac{\dot{\varphi}^2}{c^2}}}.
\end{aligned}
$$

These hideous expressions satisfy the relation

$$
g^{ab}p_a p_b = g^{tt}H^2 - 2g^{t\varphi}Hp_\varphi + g^{rr}p_r^2 + g^{\varphi\varphi}p_\varphi^2 = -m^2c^2
$$

or

$$
\begin{aligned}
&c^{-2}\frac{1 + \frac{j^2}{c^2\overline{m}^2 r} + \frac{2Gj^2}{c^4\overline{m}r^3}}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}}H^2 - \frac{2Gj}{c^4 r^3\left(1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}\right)}Hp_\varphi \\
&\quad - \left(1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}\right)p_r^2 - \frac{1}{r^2}\frac{1 - \frac{2G\overline{m}}{c^2 r}}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2\overline{m}^2 r^2}}p_\varphi^2 = m^2c^2.
\end{aligned}
$$