MA 4448
Assignment 3
Due 9 March 2011

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1. In the next problem, you will be asked to compute all non-zero components of the curvature tensor R_{abcd} for the metric whose non-zero components, in coordinates u, v, x, y, are

$$g_{uv} = g_{vu} = \frac{1}{2} \quad g_{xx} = g_{yy} = \frac{1}{4}\frac{(u+v)^2}{(1+x^2+y^2)^2}$$

To preserve your sanity and that of the person marking the assignments, you should use the definition

$$R_{debc} = g_{ae}(\Gamma^a_{bd,c} - \Gamma^a_{cd,b} + \Gamma^f_{bd}\Gamma^a_{cf} - \Gamma^f_{cd}\Gamma^a_{bf})$$

as little as possible. Which components do you need to compute from the definition, and how can you get the others?
Solution: The metric is symmetric under reversing the signs of x or y, from which it follows that

$$R_{debc} = 0$$

unless both x and y appear an even number of times in $debc$. As for any metric, the first two and last two indices must be distinct for any non-zero component, since the curvature tensor is antisymmetic in these indices. This leaves us with the list $uvuv$, $uxux$, $uxvx$, $vxvx$, $uyuy$, $uyvy$, $vyvy$ $xyxy$. The others are all zero, or can be obtained from on of these be the relations

$$R_{debc} = -R_{edbc} = -R_{decb} = R_{bcde}.$$

But we are not done yet. The metric is symmetric under switching x and y, so $uyuy$, $uyvy$, $vyvy$ are the same as $uxux$, $uxvx$, $vxvx$, except with x and y switched. The metric is also symmetric under switching u and v, so $vxvx$ is the same as $uxux$, except with u and v switched. So we are left with only $uvuv$, $uxux$, $uxvx$, $xyxy$ to compute from the definition.

2. Compute all non-zero components of the curvature tensor R_{abcd} for the metric from the previous problem.
Solution: Using the result of the previous problem, we need only compute R_{uvuv}, R_{uxux}, R_{uxvx}, and R_{xyxy} from the definition. The others are all zero or can be obtained by symmetry. First, we compute the non-zero derivatives of the metric

$$g_{xx,u} = g_{yy,u} = g_{xx,v} = g_{yy,v} = \frac{1}{2}\frac{u+v}{(1+x^2+y^2)^2}$$

$$g_{xx,x} = g_{yy,x} = -\frac{(u+v)^2 x}{(1+x^2+y^2)^3}$$

$$g_{xx,y} = g_{yy,y} = -\frac{(u+v)^2 y}{(1+x^2+y^2)^3}$$

we compute the Christoffel symbols, using the relation,

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(g_{dc,b} + g_{bd,c} - g_{cb,d} \right),$$

obtaining

$$\Gamma^u_{xx} = \Gamma^v_{xx} = \Gamma^u_{yy} = \Gamma^v_{yy} = -\frac{1}{2}\frac{u+v}{(1+x^2+y^2)^2},$$

$$\Gamma^x_{xu} = \Gamma^x_{ux} = \Gamma^x_{xv} = \Gamma^x_{vx} = \Gamma^y_{yu} = \Gamma^y_{uy} = \Gamma^y_{yv} = \Gamma^y_{vy} = \frac{1}{u+v},$$

$$\Gamma^x_{xx} = -\Gamma^x_{yy} = -\frac{x}{1+x^2+y^2} = \Gamma^y_{xy} = \Gamma^y_{yx},$$

and

$$\Gamma^y_{yy} = -\Gamma^y_{xx} = -\frac{y}{1+x^2+y^2} = \Gamma^x_{xy} = \Gamma^x_{yx}.$$

We then compute the curvature tensor using the definition

$$R_{debc} = g_{ae}(\Gamma^a_{bd,c} - \Gamma^a_{cd,b} + \Gamma^f_{bd}\Gamma^a_{cf} - \Gamma^f_{cd}\Gamma^a_{bf}).$$

R_{uvuv} is immediately seen to be zero.

$$R_{uxux} = -g_{xx}(\Gamma^x_{xu,u} + \Gamma^x_{xu}\Gamma^x_{ux}) = 0,$$

$$R_{uxvx} = g_{xx}(\Gamma^x_{xu,v} - \Gamma^x_{xu}\Gamma^x_{vx}) = 0,$$

and

$$\begin{aligned} R_{xyxy} = g_{yy}(&\Gamma^y_{xx,y} - \Gamma^y_{yx,x} + \Gamma^u_{xx}\Gamma^y_{yu} + \Gamma^v_{xx}\Gamma^y_{yv} \\ &+\Gamma^x_{xx}\Gamma^y_{yx} + \Gamma^y_{xx}\Gamma^y_{yy} - \Gamma^x_{yx}\Gamma^y_{xx} - \Gamma^y_{yx}\Gamma^y_{xy}) = 0. \end{aligned}$$

So all entries are zero. Which the spacetime is locally flat.