MA 4448
Assignment 2
Due 23 February 2011

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1. With spherical coordinates (t, r, θ, φ), let $g_{(0)} = \partial/\partial t$, $g_{(1)} = \partial/\partial r$, $g_{(2)} = r^{-1}\partial/\partial\theta$, and $g_{(3)} = (r\sin\theta)^{-1}\partial/\partial\varphi$.

   (a) Show that $\{g_{(0)}, g_{(1)}, g_{(2)}, g_{(3)}\}$ is a local frame.
   Solution: We know that $\{\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z\}$ is a basis at each point. t, r, θ, φ are related to t, x, y, z by

   $$t = t \quad x = r\sin\theta\cos\varphi \quad y = r\sin\theta\sin\varphi \quad z = r\cos\theta$$

   By the chain rule,

   $$\begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial t}{\partial t} & \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial t}{\partial r} & \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial t}{\partial \theta} & \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial t}{\partial \varphi} & \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

   $$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\varphi \\ 0 & -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

   Then

   $$\begin{pmatrix} g_{(0)} \\ g_{(1)} \\ g_{(2)} \\ g_{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^{-1} & 0 \\ 0 & 0 & 0 & (r\sin\theta)^{-1} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}$$

   $$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi & 0 \end{pmatrix}$$

has determinant 1, and hence is invertible, so $\{g_{(0)}, g_{(1)}, g_{(2)}, g_{(3)}\}$ is a basis because $\{\partial/\partial t, \partial/\partial r, \partial/\partial\theta, \partial/\partial\varphi\}$ is.

(b) Compute the metric coefficients in this frame, and show that this frame is a tetrad.

Solution: Let $\cdot$ denote inner products. Then

$$g_{(0)} \cdot g_{(0)} = \frac{\partial}{\partial t} \cdot \frac{\partial}{\partial t} = g_{tt} = -c^2,$$

$$g_{(1)} \cdot g_{(1)} = \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial r} = g_{rr} = 1,$$

$$g_{(2)} \cdot g_{(2)} = \frac{1}{r}\frac{\partial}{\partial\theta} \cdot \frac{1}{r}\frac{\partial}{\partial\theta} = \frac{1}{r^2} g_{\theta\theta} = 1,$$

and

$$g_{(3)} \cdot g_{(3)} = \frac{1}{r\sin\theta} l \frac{\partial}{\partial\varphi} \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} = \frac{1}{r^2\sin^2\theta} g_{\varphi\varphi} = 1.$$

The non-diagonal commponents are all zero. Since the components of the metric are all constant in this frame it is, by definition, a tetrad.

(c) Compute the commutation coefficients, defined by

$$[g_{(b)}, g_{(c)}] = \gamma^a_{bc} g_{(a)}.$$

Solution:

$$\begin{aligned} \left[g_{(2)}, g_{(3)}\right] = -\left[g_{(2)}, g_{(3)}\right] &= \left[\frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right] \\ &= \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) \\ &= \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi} = r^{-1}\cot\theta g_{(2)} \end{aligned}$$

so

$$\gamma^2_{23} = -\gamma^2_{32} = r^{-1}\cot\theta$$

and

$$\gamma_{23}^{a} = -\gamma_{32}^{a} = 0$$

if $a \neq 2$. The other commutators are similar, but simpler. The other non-zero components are

$$\gamma_{12}^{2} = -\gamma_{21}^{2} = -r^{-1} = \gamma_{13}^{3} = -\gamma_{31}^{3}.$$

2. Suppose $X = X^a \partial/\partial x^a$ and $Y = Y^b \partial/\partial x^b$ are vector fields and $\alpha = \alpha_c dx^c$ is a 1-form. Prove that

$$(d\alpha)(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]).$$

Solution:

By definition

$$(d\alpha)(X, Y) = (d\alpha)_{ab} X^a Y^b = \epsilon^{cd} ab \frac{\partial}{\partial x^c} \alpha_d X^a Y^b = (\alpha_{b,a} - \alpha_{a,b}) X^a Y^b.$$

Now

$$\alpha(X) = \alpha_c dx^c (X^a \partial/\partial x^a) = \alpha_c X^a dx^c (\partial/\partial x^a) = \alpha_c X^a g_a^c = \alpha_a X^a.$$

Similarly,

$$\alpha(Y) = \alpha_b Y^b.$$

Then

$$X\alpha(Y) = X^a \partial/\partial x^a (\alpha_b Y^b) = X^a (\alpha_{b,a} Y^b + \alpha_b Y^b_{,a}) = \alpha_{b,a} X^a Y^b + \alpha_b X^a Y^b_{,a}$$

and

$$Y\alpha(X) = \alpha_{a,b} X^a Y^b + \alpha_a X^a_{,b} Y^b.$$

By definition,

$$[X, Y] = XY - YX$$

$$XY = X^a \partial/\partial x^a (Y^b \partial/\partial x^b) + X^a Y^b \partial^2/\partial x^a \partial x^b$$

and

$$YX = Y^b \partial/\partial x^b (X^a \partial/\partial x^a) + X^a Y^b \partial^2/\partial x^b \partial x^a$$

so

$$[X, Y] = X^a Y^b_{,a} \partial/\partial x^b - X^a_{,b} Y^b \partial/\partial x^a$$

and

$$\alpha([X, Y]) = \alpha_c X^a Y^b_{,a} dx^c (\partial/\partial x^b) - \alpha_c X^a_{,b} Y^b dx^c (\partial/\partial x^a) = \alpha_b X^a Y^b_{,a} - \alpha_a X^a_{,b} Y^b.$$

Combining the results above

$$X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = (\alpha_{b,a} - \alpha_{a,b}) X^a Y^b.$$

This is the same as our expression for $(d\alpha)(X, Y)$.

3. Suppose $\{g_{(0)}, g_{(1)}, g_{(2)}, g_{(3)}\}$ is a general frame, *i.e.* that it's values at x are a basis for $T_x\mathcal{M}$. Let $\{g^{(0)}, g^{(1)}, g^{(2)}, g^{(3)}\}$ be the dual frame, *i.e.* its values at x are the dual basis for $T_x^*\mathcal{M} = \Omega_x^1\mathcal{M}$. Then $\{g^{(0)} \wedge g^{(1)} = -g^{(1)} \wedge g^{(0)}, g^{(0)} \wedge g^{(2)} = -g^{(2)} \wedge g^{(0)}, g^{(0)} \wedge g^{(3)} = -g^{(3)} \wedge g^{(0)}, g^{(1)} \wedge g^{(2)} = -g^{(2)} \wedge g^{(1)}, g^{(1)} \wedge g^{(3)} = -g^{(3)} \wedge g^{(1)}, g^{(2)} \wedge g^{(3)} = -g^{(3)} \wedge g^{(2)}\}$ is, at each point x, a basis for $\Omega_x^2\mathcal{M}$. What are $\{dg^{(0)}, dg^{(1)}, dg^{(2)}, dg^{(3)}\}$ in this basis?
   Note: You may use the result of the previous problem even if you didn't succeed in proving it.
   Solution:

   Apply the result of the previous problem to $X = g_{(a)}$, $Y = g_{(b)}$ and $\alpha = g^{(c)}$.

$$(dg^{(c)})(g_{(a)}, g_{(b)}) = g_{(a)}(g^{(c)}(g_{(b)})) - g_{(b)}(g^{(c)}(g_{(a)})) - g^{(c)}([g_{(a)}, g_{(b)}]).$$

   Now

$$g^{(c)}(g_{(a)}) = g_{(a)}^{(c)},$$

   which is constant, and hence has zero derivatives. Thus

$$g_{(a)}(g^{(c)}(g_{(b)})) = 0$$

   Similarly,

$$g_{(b)}(g^{(c)}(g_{(a)})) = 0$$

   By definition,

$$[g_{(a)}, g_{(b)}] = \gamma_{ab}^d g_{(d)}$$

   and hence

$$g^{(c)}([g_{(a)}, g_{(b)}]) = \gamma_{ab}^d g^{(c)}(g_{(d)}) = \gamma_{ab}^d g_{(d)}^{(c)} = \gamma_{ab}^c.$$

   Therefore

$$(dg^{(c)})(g_{(a)}, g_{(b)}) = -\gamma_{ab}^c$$

   and

$$dg^{(c)} = -\gamma_{ab}^c g^{(a)} \wedge g^{(b)}.$$