MA 4448
Assignment 1
Due 2 February 2011

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1. Figure out how the ordinary derivatives $A^{i_1\cdots i_r}_{j_1\cdots j_s,k}$ of an arbitrary (r,s) tensor $A^{i_1\cdots i_r}_{j_1\cdots j_s}$ transform under coordinate changes.
   If you get lost in the indices then can do this for a $(2,2)$, although the general case is actually easier.
   Solution: In the coordinates $\overline{x}^0$, $\overline{x}^1$, $\overline{x}^2$, $\overline{x}^3$,

$$\begin{aligned}
\overline{A}^{i_1\cdots i_r}_{j_1\cdots j_s,k} &= \frac{\partial}{\partial \overline{x}^k}\overline{A}^{i_1\cdots i_r}_{j_1\cdots j_s} \\
&= \frac{\partial x^n}{\partial \overline{x}^k}\frac{\partial}{\partial x^n}\left(\prod_{1\le p\le r}\frac{\partial \overline{x}^{i_p}}{\partial x^{l_p}}\prod_{1\le q\le s}\frac{\partial x^{m_q}}{\partial \overline{x}^{j_q}}A^{l_1\cdots l_r}_{m_1\cdots m_s}\right) \\
&= \prod_{1\le p\le r}\frac{\partial \overline{x}^{i_p}}{\partial x^{l_p}}\prod_{1\le q\le s}\frac{\partial x^{m_q}}{\partial \overline{x}^{j_q}}\frac{\partial x^n}{\partial \overline{x}^k}\frac{\partial}{\partial x^n}A^{l_1\cdots l_r}_{m_1\cdots m_s} \\
&\quad + \sum_{1\le u\le r}\prod_{\substack{1\le p\le r\\ p\ne u}}\frac{\partial \overline{x}^{i_p}}{\partial x^{l_p}}\prod_{1\le q\le s}\frac{\partial x^{m_q}}{\partial \overline{x}^{j_q}}\frac{\partial x^n}{\partial \overline{x}^k}\frac{\partial^2 \overline{x}^{i_u}}{\partial x^n\partial x^{l_u}}A^{l_1\cdots l_r}_{m_1\cdots m_s} \\
&\quad + \sum_{1\le v\le s}\prod_{1\le p\le r}\frac{\partial \overline{x}^{i_p}}{\partial x^{l_p}}\prod_{\substack{1\le q\le s\\ q\ne v}}\frac{\partial x^{m_q}}{\partial \overline{x}^{j_q}}\frac{\partial x^n}{\partial \overline{x}^k}\frac{\partial^2 x^{m_v}}{\partial x^n\partial \overline{x}^{j_v}}A^{l_1\cdots l_r}_{m_1\cdots m_s}.
\end{aligned}$$

2. Compute the Christoffel symbols of Minkowski space in

   (a) Cartesian coordinates: $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. For your sanity and mine, write indices as t, x, y and z rather than 0, 1, 2 and 3. So the metric is $g_{tt} = -c^2$, $g_{xx} = g_{yy} = g_{zz} = 1$ and all other components are zero.
   Solution: Every derivative of every metric component is zero, so all the connection coefficients are zero.

(b) Spherical Coordinates: $(x^0, x^1, x^2, x^3)(t, r, \theta, \varphi)$. The metric in these coordinates is $g_{tt} = -c^2$, $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2 \sin^2\theta$ and all other components are zero.
Solution:

The inverse metric is easily computed because the matrix representation of the metric is diagonal in this basis.

$$g^{tt} = -c^{-2} \quad g^{rr} = 1 \quad g^{\theta\theta} = r^{-2} \quad g^{\varphi\varphi} = r^{-2}\csc^2\theta$$

The non-zero derivatives of the metric components are

$$g_{\theta\theta,r} = 2r \quad g_{\varphi\varphi,r} = 2r\sin^2\theta \quad g_{\varphi\varphi,\theta} = 2r^2\sin\theta\cos\theta.$$

Using the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(g_{kl,j} + g_{jl,k} - g_{jk,l}\right)$$

we see that, of the 64 entries of the Christoffel symbol, the only non-zero ones are

$$\Gamma^r_{\theta\theta} = -r \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = r^{-1} = \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r}$$

$$\Gamma^r_{\varphi\varphi} = -r\sin^2\theta \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \quad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

3. Read about raising and lowering of indices in Section 3.1.1 of George Ellis' notes. It's fairly simple. Then,

(a) Show that $g^{ij}_{;k}$ is zero.
Solution:

There is no need to deal with Christoffel symbols. We already know that the covariant derivative of g with two lower indices is zero. That was part of the definition of the covariant derivative. We also know that the covariant derivative g with an upper and lower index is zero. That was proved in lecture. So,

$$0 = g^i_{l;k} = (g^{im} g_{lm})_{;k} = g^{im}_{;k} g_{lm} + g^{im} g_{lm;k} = g^{im}_{;k} g_{lm}.$$

Contract with g^{lj},

$$0 = g^{im}_{;k} g_{lm} g^{jl} = g^{im}_{;k} g^j_m = g^{ij}_{;k}.$$

(b) What do raising and lowering correspond to in the coordinate-free description of tensors?
Solution: The metric tensor is a non degenerate quadratic form on T_xM. It gives us a isomorphism between T_xM and T_x^*M as follows. Take a vector in T_xM. Taking the quadratic form applied to this vector and some other vector, we get something linear in the second vector. In other words, we get an element of T_x^*M. This element depends linearly on the first vector. So we have a linear map from T_xM to T_x^*M. The fact that this has zero nullspace follows immediately from non-degeneracy. The vector spaces have the same dimension, so an injective linear function is an isomorphism. Lowering an index corresponds to applying this isomorphism to one of the T_xM factors in the tensor product. Raising an index corresponds to applying its inverse to one of the T_x^*M factors.

(c) Prove that we get the same result from differentiating and then raising/lowering as we get from raising/lowering and then differentiating. Differentiating here means covariant differentiation.
Solution: There is no need to mention Christoffel symbols at all. We raise an index by contracting with g^{ij} and lower by contracting with g_{ij}. Applying the Leibnitz rule, we get two terms. The first is g contracted with the covariant derivative of our tensor. That's what we want. The second term is zero, because all covariant derivatives of g are zero, whether it has upper or lower indices.