MA 4448
Assignment 4
Due 31 March 2011

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1. The Kerr metric is, in coordinates t, r, θ, φ,

$$g_{tt} = -c^2 \left(1 - \frac{2G\overline{m}r}{c^2r^2\left(1 + \frac{j^2\cos^2\theta}{\overline{m}^2c^2r^2}\right)}\right)$$

$$g_{t\varphi} = g_{\varphi t} = -\frac{2Gj\sin^2\theta}{c^2r\left(1 + \frac{j^2\cos^2\theta}{\overline{m}^2c^2r^2}\right)}$$

$$g_{rr} = \frac{1 + \frac{j^2\cos^2\theta}{c^2\overline{m}^2r^2}}{1 - \frac{2G\overline{m}}{c^2r} + \frac{j^2}{c^2\overline{m}^2r^2}}$$

$$g_{\theta\theta} = r^2\left(1 + \frac{j^2\cos^2\theta}{\overline{m}^2c^2r^2}\right)$$

$$g_{\varphi\varphi} = \left(r^2 + \frac{j^2}{c^2\overline{m}^2} + \frac{2Gj^2\sin^2\theta}{c^4\overline{m}r\left(1 + \frac{j^2\cos^2\theta}{\overline{m}^2c^2r^2}\right)}\right)\sin^2\theta.$$

Here J is the angular momentum of the central mass and $\overline{m}$ is its mass. G is the Newtonian gravitational constant. All other entries of the metric tensor are zero. Find the redshift between an arbitrary pair of stationary observers.
Note: Don't be frightened by the hideousness of the metric. The problem is in fact really easy.

2. For the Kerr metric, as for the Reissner-Nordström, $\theta \to \pi - \theta$ is a symmetry, so if $\theta = \pi/2$ initially and $\dot{\theta} = 0$ then $\theta = \pi/2$ for all time. We *don't* have full rotational symmetry so we can *not* say that $\theta = \pi/2$ without loss of generality. But the case $\theta = \pi/2$ is still physically interesting. This problem is equivalent to studying the motion of a particle in a three dimensional space time with coordinates t, r and φ with metric tensor obtained by setting $\theta = \pi/2$ in the equations above and then dropping $g_{\theta\theta}$:

$$g_{tt} = -c^2 \left(1 - \frac{2G\overline{m}}{c^2 r}\right)$$

$$g_{t\varphi} = g_{\varphi t} = -\frac{2Gj}{c^2 r}$$

$$g_{rr} = \frac{1}{1 - \frac{2G\overline{m}}{c^2 r} + \frac{j^2}{c^2 \overline{m}^2 r^2}}$$

$$g_{\varphi\varphi} = \left(r^2 + \frac{j^2}{c^2 \overline{m}^2} + \frac{2Gj^2}{c^4 \overline{m} r}\right).$$

What is the inverse metric?

3. What are the canonical momenta and Hamiltonian for an uncharged test particle of mass m on a spacetime with metric as in the previous problem? What algebraic relation do they satisfy?