Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
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Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

JS Maths **Semester 2 2025-2026**
SS Maths

MAU34804 Fixed Point Theorems and Economic Equilibria

24 April 2026 **Goldsmith** **17.00–19.00**

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Instructions to Candidates:

Calculators or mathematical tables are permitted, but unlikely to be helpful.

Instructions for Invigilators:

Credit will be given for the best 3 questions answered.

1. (20 points) The following theorem was not stated or proved in lecture or in the notes.

Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, let $f\colon X \times Y \to \mathbf{R}$ be a continuous real-valued strictly concave function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a non-empty valued convex valued correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is non-empty, convex, and compact for all $\mathbf{x} \in X$ and that the correspondence $\Phi\colon X \rightrightarrows Y$ is both upper hemicontinuous and lower hemicontinuous. Let

$$m(\mathbf{x}) = \max\{f(\mathbf{x}, \mathbf{y})\colon \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x})\colon f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then $m\colon X \to \mathbf{R}$ is continuous, and there is a continuous function $\mathbf{r}\colon X \to Y$ such that

$$M(\mathbf{x}) = \{\mathbf{r}(\mathbf{x})\}$$

for all $x \in X$.

(a) (5 points) This theorem is similar to the Berge Maximum Theorem, which was stated and proved in the notes. In what ways do the two theorems differ?

Solution: There are two additional hypotheses, that $\Phi(\mathbf{x})$ is convex and that f is strictly concave. Also, the conclusion that M is of the form

$$M(\mathbf{x}) = \{\mathbf{r}(\mathbf{x})\}$$

is new.

Comments: Technically the fact that M is non-empty valued and upper hemicontinuous M has been dropped from the conclusion, the first part is obvious because $\{\mathbf{r}(\mathbf{x})\} \neq \varnothing$ and the second part follows from the continuity of $\mathbf{r}$ by a theorem in the notes, so it's okay not to mention the first of these changes and I might overlook omitting the second.

(b) (5 points) Give an example to show that the conclusion does not hold if we drop the convexity hypothesis.

Solution: If we drop the convexity hypothesis then we're back in the setting of Berge's maximum theorem so any example where M does not come from a function will do. The example given in lecture was $X = Y = [-1, 1]$, $\Phi(x) = Y$, and $f(x, y) = xy$.

Comments: Lots of examples will work.

(c) (10 points) Prove the theorem. You may use the Berge Maximum Theorem and any other results from the notes or lecture.

Solution: The hypotheses include all the hypotheses of the Berge maximum theorem so we get the conclusion, that is continuity of m and upper hemicontinity of M. Suppose $\mathbf{y}, \mathbf{z} \in M(\mathbf{x})$ for some $\mathbf{x} \in X$. Then $\frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z} \in Y$ so

$$f\left(\frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}\right) \leq m(\mathbf{x})$$

since $m(\mathbf{x})$ is the maximum. But if $\mathbf{y} \neq \mathbf{z}$ then

$$f\left(\frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}\right) > \frac{1}{2}f(\mathbf{y}) + \frac{1}{2}f(\mathbf{z}) = m(\mathbf{x})$$

by strict concavity. This is a contradiction, so there is at most one $\mathbf{y} \in M(\mathbf{x})$. From Berge we know that there is at least one so there's exactly one and we can define $\mathbf{r}(\mathbf{x})$ to be this $\mathbf{y}$. A theorem in the notes then shows that $\mathbf{r}$ is continuous since M is upper hemicontinuous.

Comments: This is rather tricky, but the argument is essentially the one which was used to construct a retraction from $\mathbf{R}^n$ to a non-empty closed subset in lecture.

2. (20 points)

(a) (2 points) What is the interior of a simplex, as defined in the notes and lecture?

Solution: The interior of a simplex was defined to be the set of points which don't belong to a proper face.

Comments: The interior is also the set where all barycentric coordinates are positive. That's not the actual definition but I'd accept it.

(b) (2 points) What is the topological interior of a simplex, as defined in the notes and lecture?

Solution: The topological interior is the set of points in the simplex such that there's some positive r such that the ball of radius r about the point is also in the simplex.

(c) (4 points) What are the interior and topological interior of the simplex in $\mathbf{R}^2$ with vertices $(0,1)$, $(1,1)$ and $(1,0)$?

Solution: The interior is the set of points of the form

$$s_0(0,1) + s_1(1,1) + s_2(1,0) = (s_1 + s_2, s_0 + s_1)$$

with $s_0 > 0$, $s_1 > 0$, $s_2 > 0$ and $s_0 + s_1 + s_2 = 1$. The topological interior is the same as the interior.

Comments: It would also be fine to describe the interior as the set of $(x, y) \in \mathbf{R}^2$ such that $x < 1$, $y < 1$, and $x + y > 1$.

(d) (4 points) What are the interior and topological interior of the simplex in $\mathbf{R}^2$ with vertices $(0,1)$ and $(1,0)$?

Solution: The interior is the set of points of the form $s_0(0,1) + s_1(1,0) = (s_1, s_0)$ with $s_0 > 0$, $s_1 > 0$, and $s_0 + s_1 = 1$. The topological interior is empty.

Comments: It would also be fine to describe the interior as the set of $(x, y) \in \mathbf{R}^2$ such that $x > 0$, $y > 0$, and $x + y = 1$.

(e) (8 points) Prove that the interior of a k-simplex in $\mathbf{R}^n$ is equal to its topological interior if and only if $k = n$.

Solution: The relative interior of the simplex is the set of points

$$\mathbf{v}_0 + \sum_{j=1}^{n} r_j \left(\mathbf{v}_j - \mathbf{v}_{j-1}\right)$$

where the vertices of the simplex are $\mathbf{v}_0$, ..., $\mathbf{v}_k$ and $r_1 < r_2 < \cdots < r_k$. For any simplex the vectors $\mathbf{v}_j - \mathbf{v}_{j-1}$ are linearly independent.

Suppose first that $k = n$. Then the vectors $\mathbf{v}_j - \mathbf{v}_{j-1}$ form a basis so for every $\mathbf{w}$ in $\mathbf{R}^n$ there are unique numbers $r_1, \ldots, r_n$ such that

$$\mathbf{w} - \mathbf{v}_0 = \sum_{j=1}^{n} r_j \left(\mathbf{v}_j - \mathbf{v}_{j-1}\right).$$

In other words, all vectors are of the form

$$\mathbf{v}_0 + \sum_{j=1}^{n} r_j \left(\mathbf{v}_j - \mathbf{v}_{j-1}\right)$$

and the ones in the relative interior are just the ones satisfying the additional requirement that $r_1 < r_2 < \cdots < r_k$. Near any such point all sufficiently nearby points satisfy the same condition so the relative interior is open and therefore is the interior.

On the other hand, if $k < n$ then the vectors $\mathbf{v}_j - \mathbf{v}_{j-1}$ are linearly independent but do not form a basis so they don't span. In other words there is a vector $\mathbf{u}$ which cannot be written as a linear combination of them. If we look at the points

$$\mathbf{v}_0 + \sum_{j=1}^{n} \mathbf{su} + r_j \left(\mathbf{v}_j - \mathbf{v}_{j-1}\right)$$

then they do not lie in the simplex when $s \neq 0$. We can find such points arbitrarily close to

$$\mathbf{v}_0 + \sum_{j=1}^{n} r_j \left(\mathbf{v}_j - \mathbf{v}_{j-1}\right)$$

so the interior of the simplex is empty, and therefore not the same as the relative interior.

Comments: This proof uses some basic facts from linear algebra, e.g. that any two bases have the same number of elements or that the coordinates of a vector in a real vector space with respect to a basis depend continuously on the vector. I wouldn't expect you to prove any of those, or even necessarily state them, as long as I can follow your argument.

3. (20 points) The statement of the Kakutani Fixed Point Theorem is

> Suppose $X \subseteq \mathbf{R}^n$ is a non-empty compact convex subset of a Euclidean space and $\Phi \colon X \rightrightarrows X$ is non-empty valued, convex valued and has closed graph. Then Φ has a fixed point, i.e. there is an $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

(a) (4 points) Give an example to show that the conclusion does not hold if we drop the compactness hypothesis.

Solution: We can take $X = \mathbf{R}^n$ and $\Phi(\mathbf{x}) = \{\mathbf{x} + \mathbf{a}\}$ for some $\mathbf{a} \neq \mathbf{0}$.

Comments: This is essentially the counterexample for Brouwer given in lecture converted into a counterexample for Kakutani by constructing a correspondence from a function in the usual way.

Lots of other examples will work, of course.

(b) (4 points) Give an example to show that the conclusion does not hold if we drop the convexity hypothesis.

Solution: We can take X to be the unit sphere and Φ to be the correspondence $\Phi(\mathbf{x}) = \{-\mathbf{x}\}$.

Comments: Again, this is a counterexample for Brouwer from the lecture converted to a counterexample for Kakatani.

(c) (5 points) In what ways does the statement of the Kakutani Fixed Point Theorem differ from the statement of the Brouwer Fixed Point Theorem?

Solution: Brouwer doesn't require compactness or convexity but does require the X is homeomorphic to a closed ball. Also Brouwer has a continuous function instead of a non-empty valued convex valued correspondence with closed graph. Also, the meaning of fixed point for a function is $\mathbf{x}^* = f(\mathbf{x}^*)$ while for a correspondence it's

$$\mathbf{x}^* \in \Phi(\mathbf{x}^*)$$

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(d) (7 points) Deduce the Brouwer Fixed Point Theorem from the Kakutani Fixed Point Theorem. In lecture and in the notes Kakutani was deduced from Brouwer, so this is technically circular, but don't worry about it. In addition to the Kakutani Fixed Point Theorem you may use any results from the notes or lecture other than the Brouwer Fixed Point Theorem itself.

Solution: Suppose X is homeomorphic to a closed ball and $f\colon X \to X$ is contionous, as in the hypotheses of the Brouwer fixed point theorem. By assumption there is a homemorphism φ from the a closed ball to X. Then $g = \varphi \circ f \circ \varphi^{-1}$ is a continuous function from the ball to itself and $\mathbf{x}^*$ is a fixed point of f if and only if $\mathbf{y}^* = \varphi(\mathbf{x}^*)$ is a fixed point of g. So we just need to show that continuous functions from a closed ball to itself have a fixed point.

Suppose then that g is a continuous function from a closed ball to itself and define a correspondence Φ by $\Phi(\mathbf{y}) = \{g(\mathbf{y})\}$. For each $\mathbf{y}$ the set $\Phi(\mathbf{y})$ is non-empty and convex, so we just need to check that the graph of Φ is closed, but closed balls are compact to this follows from the upper hemicontinuity of Φ by a theorem in the notes, and that follows from the continuity of g by another theorem in the notes. By the Kakatani fixed point theorem Φ has a fixed point, i.e. a point $\mathbf{y}^*$ such that $\mathbf{y}^* \in \Phi(\mathbf{y}^*)$. But $\Phi(\mathbf{y}^*) = \{g(\mathbf{y}^*)\}$ so $\mathbf{y}^* = g(\mathbf{y}^*)$. In other words, $\mathbf{y}^*$ is a fixed point of g.

Comments: The first part of the argument, reducing to the case where X is a closed ball was given in lecture but it was given as part of the proof of Brouwer, which is what we're proving, so it should be repeated here, but I wouldn't be too picky about that.

4. (20 points)

(a) (4 points) What hypotheses are utility functions required to satisfy for the theorems on general equilibria in the notes?

Solution:

- continuous
- strictly increasing, i.e. if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+$ and $\mathbf{x} \leq \mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$ then $u(\mathbf{x}) < u(\mathbf{y})$.
- quasiconcave, i.e. if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+$ and $0 \leq t \leq 1$ then

$$u((1-t)\mathbf{x} + t\mathbf{y}) \geq \min\{u(\mathbf{x}), u(\mathbf{y})\}.$$

(b) (16 points) Of the following four functions, exactly one is a utility function. Which one is it? Check that it satisfies all the conditions. For the other three, indicate which condition or conditions fail.

i.

$$u(p,q) = \begin{cases} 6p + 3q & \text{if } 2p < 3q \\ 2p + 9q & \text{if } 2p \geq 3q \end{cases}$$

ii.

$$u(p,q) = \begin{cases} 2p + 6q & \text{if } 2p < 3q \\ 4p + 3q & \text{if } 2p \geq 3q \end{cases}$$

iii.

$$u(p,q) = \begin{cases} 2p + 6q & \text{if } 2p < 3q \\ 2p + 3q & \text{if } 2p \geq 3q \end{cases}$$

iv.

$$u(p,q) = \begin{cases} 6p - 3q & \text{if } 2p < 3q \\ -2p + 9q & \text{if } 2p \geq 3q \end{cases}$$

Solution: First we check continuity. Subtracting off a linear function won't affect continuity so we can subtract off the linear function used to define the function in the $2p < 3q$ region everywhere and look at the remaining function, which will be equal to 0 when $2p < 3q$ and will be equal to $-4p + 6q$, $2p - 3q$, $-3q$, or

$-8q + 12p$ in the region $2p \geq 3q$, depending on which function we're looking at. All but the third one are multiples of $2p - 3q$ and so vanish on the line $2p - 3q = 0$ which divides the regions and so the values are the same as we approach from either side, as required. In other words, only the third one is discontinuous.

Next we check which ones are strictly increasing. This is a bit trickier than it looks. The fourth function is clearly not strictly increasing because increasing q in the first region or p in the second region decreases the value of $u(p, q)$. The first and second functions are strictly increasing, since increasing p or q in either region increases $u(p, q)$ and there is no jump at the interface between the regions because of the continuity of u. To see that we do need this last observation, consider the third function. At first sight it looks like it should be strictly increasing but consider increasing q for a fixed positive value of p, starting from the second region. As we cross the boundary between the regions the value of u suddenly drops by $3q = 2p > 0$, so the third function is not strictly increasing.

Finally we check quasiconcavity. By a lemma from lecture u is quasiconcave if the set $u^*([a, \infty))$ of points where $u(p, q) \geq 0$ is convex. For each function the various values of a all give congruent sets so it suffices to check $a = 1$. For the first function we have everything above and to the right of the curve consisting of a line segment from $(0, 1/3)$ to $(1/8, 1/12)$ and then the line segment from there to $(1/2, 0)$. The point $(1/8, 1/12)$ lies below the line joining $(0, 1/3)$ and $(1/2, 0)$ so this set is convex. The calculation for the second function is similar, but the points are $(0, 1/6)$, $(1/6, 1/9)$ and $(1/4, 0)$. This time the middle point lies above the line joining the first and last point, so this set is not convex. The third function, as usual, is weird, with three line segments, joining the four points $(0, 1/6)$, $(1/8, 1/12)$, $(1, 6, 1/9)$ and $(1/2, 0)$. The region above and to the right of this is not convex. For the fourth function $(u^*([1, +\infty))$ is the wedge between the lines $6p - 3q = 1$ and $-2p + 9q = 1$, which is convex.

So the only one which is a utility function is the first one. The second in not quasiconcave, the last one isn't strictly increasing, and the third one isn't anything.

Comments: There are other ways to do this.