Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

JS Maths **Semester 2 2025-2026**
SS Maths

MAU34804 Fixed Point Theorems and Economic Equilibria

24 April 2026 **Goldsmith** **17.00–19.00**

Prof. John Stalker

Instructions to Candidates:

Calculators or mathematical tables are permitted, but unlikely to be helpful.

Instructions for Invigilators:

Credit will be given for the best 3 questions answered.

1. (20 points) The following theorem was not stated or proved in lecture or in the notes.

   Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, let $f\colon X \times Y \to \mathbf{R}$ be a continuous real-valued strictly concave function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a non-empty valued convex valued correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is non-empty, convex, and compact for all $\mathbf{x} \in X$ and that the correspondence $\Phi\colon X \rightrightarrows Y$ is both upper hemicontinuous and lower hemicontinuous. Let

   $$m(\mathbf{x}) = \max\{f(\mathbf{x}, \mathbf{y})\colon \mathbf{y} \in \Phi(\mathbf{x})\}$$

   for all $\mathbf{x} \in X$, and let

   $$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x})\colon f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

   for all $\mathbf{x} \in X$. Then $m\colon X \to \mathbf{R}$ is continuous, and there is a continuous function $\mathbf{r}\colon X \to Y$ such that

   $$M(\mathbf{x}) = \{\mathbf{r}(\mathbf{x})\}$$

   for all $x \in X$.

   (a) (5 points) This theorem is similar to the Berge Maximum Theorem, which was stated and proved in the notes. In what ways do the two theorems differ?

   (b) (5 points) Give an example to show that the conclusion does not hold if we drop the convexity hypothesis.

   (c) (10 points) Prove the theorem. You may use the Berge Maximum Theorem and any other results from the notes or lecture.

2. (20 points)

   (a) (2 points) What is the interior of a simplex, as defined in the notes and lecture?

   (b) (2 points) What is the topological interior of a simplex, as defined in the notes and lecture?

   (c) (4 points) What are the interior and topological interior of the simplex in $\mathbf{R}^2$ with vertices $(0,1)$, $(1,1)$ and $(1,0)$?

   (d) (4 points) What are the interior and topological interior of the simplex in $\mathbf{R}^2$ with vertices $(0,1)$ and $(1,0)$?

   (e) (8 points) Prove that the interior of a k-simplex in $\mathbf{R}^n$ is equal to its topological interior if and only if $k = n$.

3. (20 points) The statement of the Kakutani Fixed Point Theorem is

> Suppose $X \subseteq \mathbf{R}^n$ is a non-empty compact convex subset of a Euclidean space and $\Phi\colon X \rightrightarrows X$ is non-empty valued, convex valued and has closed graph. Then Φ has a fixed point, i.e. there is an $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

(a) (4 points) Give an example to show that the conclusion does not hold if we drop the compactness hypothesis.

(b) (4 points) Give an example to show that the conclusion does not hold if we drop the convexity hypothesis.

(c) (5 points) In what ways does the statement of the Kakutani Fixed Point Theorem differ from the statement of the Brouwer Fixed Point Theorem?

(d) (7 points) Deduce the Brouwer Fixed Point Theorem from the Kakutani Fixed Point Theorem. In lecture and in the notes Kakutani was deduced from Brouwer, so this is technically circular, but don't worry about it. In addition to the Kakutani Fixed Point Theorem you may use any results from the notes or lecture other than the Brouwer Fixed Point Theorem itself.

4. (20 points)

   (a) (4 points) What hypotheses are utility functions required to satisfy for the theorems on general equilibria in the notes?

   (b) (16 points) Of the following four functions, exactly one is a utility function. Which one is it? Check that it satisfies all the conditions. For the other three, indicate which condition or conditions fail.

      i.
$$u(p,q) = \begin{cases} 6p + 3q & \text{if } 2p < 3q \\ 2p + 9q & \text{if } 2p \geq 3q \end{cases}$$

      ii.
$$u(p,q) = \begin{cases} 2p + 6q & \text{if } 2p < 3q \\ 4p + 3q & \text{if } 2p \geq 3q \end{cases}$$

      iii.
$$u(p,q) = \begin{cases} 2p + 6q & \text{if } 2p < 3q \\ 2p + 3q & \text{if } 2p \geq 3q \end{cases}$$

      iv.
$$u(p,q) = \begin{cases} 6p - 3q & \text{if } 2p < 3q \\ -2p + 9q & \text{if } 2p \geq 3q \end{cases}$$