MAU34804

Lecture 23

2026-03-25

Plan for next week

We'll finish the mathematical (and economic) content of the module today. As usual for sophister maths modules we don't have classes in the final week of term. What about next week?

- Monday: review of module content, information about exam and how to revise for it.
- Tuesday: question and answer session[1]
- Wednesday: no class, unless we ran out of time on Tuesday

[1] The number of answers will be at most as large as the number of questions.

Proposition 7.8

If household h has an initial endowment of $\overline{\mathbf{x}}_h$ and the prevailing prices are $\mathbf{p}$ then they can sell that endowment for a wealth of $\mathbf{p} \cdot \overline{\mathbf{x}}_h$ and buy whatever bundle of goods maximises their utility for that price/wealth pair.

Proposition 7.8 *Suppose the* $\overline{\mathbf{x}}_h \gg \mathbf{0}$, $\mathbf{c} \gg \mathbf{0}$. *Suppose further that* $u_h \colon \mathbf{R}^n_+ \to \mathbf{R}$ *is quasiconcave, strictly increasing and continuous. Let* $\Delta \subseteq \mathbf{R}^n_+$ *and* $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbf{R}^n_+$ *be defined by*

$$\Delta = \left\{ \mathbf{p} \in \mathbf{R}^n_+ \colon \sum_{j=1}^{n} p_i = 1 \right\},$$

$$\hat{V}_{\mathbf{c},h}(\mathbf{p}) = \max_{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, \mathbf{p} \cdot \overline{\mathbf{x}}_h)} u_h(\mathbf{x}),$$

$$\hat{\xi}_{\mathbf{c},h}(\mathbf{p}) = \left\{ \mathbf{x} \in B_{\mathbf{c}}\left(\mathbf{p}, \mathbf{p} \cdot \overline{\mathbf{x}}_h\right) : u_h(\mathbf{x}) = \hat{V}_{\mathbf{c},h}(\mathbf{p}) \right\}$$

Then $\hat{\xi}_{\mathbf{c},h}$ *is non-empty valued, compact valued, convex valued and upper hemicontinuous.*

Comments on Proposition 7.8

- We've assumed that $\bar{\mathbf{x}}_h \gg \mathbf{0}$. That was purely to ensure that the initial wealth of the household is not zero, even if some goods are free.
This assumption can be weakened.
- We've also assumed rationing with quantity limits $\mathbf{c} \gg \mathbf{0}$. This assumption prevents us from "buying" unlimited quantities of free goods.
- The assumption that $\sum_{i=1}^{n} p_i = 1$ looks unnatural, and is.
It's harmless though, as long as there is at least one non-free good, and the quantities bought and sold by a utility maximising household are unchanged if all prices are multiplied by a positive constant.
It's technically useful, because it restricts prices to a compact set.

Corollary 7.9

Corollary 7.9 *With notation and hypotheses as in Proposition 7.8,* $\mathbf{d}_c \colon \Delta \rightrightarrows \mathbf{R}^n_+$*, defined by*

$$\mathbf{d}_c = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h},$$

is non-empty valued, compact valued, convex valued and upper hemicontinuous. Furthermore, $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{s}$ *for all* $\mathbf{x} \in \mathbf{d}_\mathbf{c}(\mathbf{p})$*, where*

$$\mathbf{s} = \sum_{h=1}^{n} \overline{\mathbf{x}}_h.$$

The economic interpretation of $\mathbf{d}_\mathbf{c}(\mathbf{p})$ is as the *aggregate demand* at prices $\mathbf{p}$ subject to rationing constraints $\mathbf{c}$.

The economic interpretation of $\mathbf{s}$ is as the *aggregate supply*. This is a pure exchange economy, without production, so supply is independent of price.

Theorem 7.10

The following general theorem will be used to deduce the existence of equilibria in exchange economies.

Theorem 7.10 *Suppose* $K \subset \mathbf{R}^n$ *is compact and* $\zeta\colon \Delta \rightrightarrows K$ *is non-empty valued, closed valued, convex valued and upper hemicontinuous. If* $\mathbf{p} \cdot \mathbf{z} \leq 0$ *for all* $\mathbf{p} \in \Delta$ *and* $\mathbf{z} \in \zeta(\mathbf{p})$ *then there is a* $\mathbf{p}^* \in \Delta$ *and a* $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$ *such that* $\mathbf{z}^* \leq \mathbf{0}$.

Later we will apply this to the *excess demand* correspondence,
$\zeta(\mathbf{p}) = \{\mathbf{x} \in \mathbf{R}^n\colon \mathbf{x} + \mathbf{s} \in \xi(\mathbf{p})\}$.

Choose a compact convex set L containing K, e.g. a large closed ball.

Let $\gamma(\mathbf{x}) = \max_{1 \leq i \leq n} x_i$ and

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta\colon \mathbf{p} \cdot \mathbf{x} = \gamma(\mathbf{x})\},$$

as in Proposition 7.3.

We saw there that μ is non-empty valued, compact valued, convex valued and upper hemicontinuous.

Also $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p}' \cdot \mathbf{x}$ for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$.

Proof of Theorem 7.10, continued

Define $\Phi\colon \Delta \times L \rightrightarrows \Delta \times L$ by

$$\Phi(\mathbf{p}, \mathbf{z}) = (\mu(\mathbf{z}), \zeta(\mathbf{p})).$$

μ and ζ are closed valued and upper hemicontinuous, so they have closed graphs by Proposition 2.11.
Φ therefore is convex valued and has closed graph.
By the Kakutani Fixed Point Theorem there is a $(\mathbf{p}^*, \mathbf{z}^*) \in \Delta \times L$ such that $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$.
Now $\mathbf{p}^* \cdot \mathbf{z} \leq 0$ for all $\mathbf{z} \in \zeta(\mathbf{p}^*)$.
Also $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ implies $\mathbf{p} \cdot \mathbf{z}^* \leq \mathbf{p}^* \cdot \mathbf{z}$ for all $\mathbf{p} \in \Delta$.
So $\mathbf{p} \cdot \mathbf{z}^* \leq 0$ for all $\mathbf{p} \in \Delta$.
This is only possible if $\mathbf{z}^* \leq \mathbf{0}$.

Main theorem on exchange economies

Theorem 7.11 *Suppose that* $\overline{x}_{hi} > 0$ *for* $1 \leq h \leq m$ *and* $1 \leq i \leq n$ *and that* $u_h \colon \mathbf{R}^n_+ \to \mathbf{R}$ *is continuous, strictly increasing and quasiconcave for* $1 \leq h \leq m$*. Then there are positive* p^*_i *for* $1 \leq i \leq n$ *and non-negative* x^*_{hi} *for* $1 \leq h \leq m$ *and* $1 \leq i \leq n$ *such that*

$$\sum_{i=1}^{n} p^*_i x^*_{hi} = \sum_{i=1}^{n} p^*_i \overline{x}_{hi}$$

for $1 \leq h \leq m$ *and*

$$\sum_{h=1}^{m} x^*_{hi} = \sum_{h=1}^{m} \overline{x}_{hi}$$

for $1 \leq i \leq n$*. Also, for any* $1 \leq h \leq m$ *and any* $(x_1, \ldots, x_n) \in \mathbf{R}^n_+$*,*

$$\sum_{i=1}^{n} p^*_i x_i \leq \sum_{i=1}^{n} p^*_i \overline{x}_{hi} \Rightarrow u_h(x_1, \ldots x_n) \leq u_h\left(x^*_{h1}, \ldots, x^*_{hn}\right).$$

Economic interpretation

- $\overline{x}_{hi}$ is the amount of good i initially held by household h.
- u_h is the utility function for household h.
- p_i^* is the market clearing price of good i.
- x_{hi}^* is the amount of good i held by household h after redistribution.
- The equation $\sum_{i=1}^n p_i^* x_{hi}^* = \sum_{i=1}^n p_i^* \overline{x}_{hi}$ expresses the fact that each household breaks even.
- The equation $\sum_{h=1}^m x_{hi}^* = \sum_{h=1}^m \overline{x}_{hi}$ expresses the fact that goods are conserved.
- The implication

$$\sum_{i=1}^{n} p_i^* x_i \leq \sum_{i=1}^{n} p_i^* \overline{x}_{hi} \Rightarrow u_h(x_1, \ldots x_n) \leq u_h\left(x_{h1}^*, \ldots, x_{hn}^*\right)$$

expresses the fact that each household's utility is maximised, subject to its budget constraint.

Proof of Theorem 7.11

Let $\bar{\mathbf{x}}_h = (\bar{x}_{h1}, \ldots, \bar{x}_{hn})$, $\mathbf{x}_h^* = (x_{h1}^*, \ldots, x_{hn}^*)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)$,

$$\Delta = \left\{ \mathbf{p} \in \mathbf{R}_+^n \colon \sum_{i=1}^n p_i = 1 \right\},, \quad B_{\mathbf{c}}(\mathbf{p}, w) = \left\{ \mathbf{x} \in \mathbf{R}_+^n \colon \mathbf{x} \leq \mathbf{c}, \mathbf{p} \cdot \mathbf{x} \leq w \right\},$$

$$\hat{V}_{\mathbf{c},h}(\mathbf{p}) = \max_{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, \mathbf{p} \cdot \bar{\mathbf{x}}_h)} u_h(\mathbf{x}) \quad \hat{\xi}_{\mathbf{c},h}(\mathbf{p}) = \left\{ \mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, \mathbf{p} \cdot \bar{\mathbf{x}}_h) \colon u_h(\mathbf{x}) = \hat{V}_{\mathbf{c},h}(\mathbf{p}) \right\},$$

$$\mathbf{d}_{\mathbf{c}} = \sum_{h=1}^m \hat{\xi}_{\mathbf{c},h}, \quad \mathbf{s} = \sum_{h=1}^m \mathbf{x}_h.$$

By Proposition 7.9 $\mathbf{d}_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbf{R}_+^n$ is non-empty valued, compact valued, convex valued and upper hemicontinuous if $\mathbf{c} \gg \mathbf{0}$.
Also, $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{s}$ if $\mathbf{x} \in \mathbf{d}_{\mathbf{c}}(\mathbf{p})$.

Proof of Theorem 7.11 continued

Define $\zeta_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbf{R}^n$ by

$$\zeta_{\mathbf{c}}(\mathbf{p}) = \{\mathbf{z} \in \mathbf{R}^n \colon \mathbf{z} + \mathbf{s} \in \mathbf{d}_{\mathbf{c}}(\mathbf{p})\}\,.$$

Then $\mathbf{p} \cdot \mathbf{z} \leq 0$ for all $\mathbf{z} \in \zeta_{\mathbf{c}}(\mathbf{p})$.
$\zeta_{\mathbf{c}}(\Delta) \subseteq \{\mathbf{z} \in \mathbf{R}^n \colon -\mathbf{s} \leq \mathbf{z} \leq \mathbf{c} - \mathbf{s}\}$, which is compact, so by Theorem 7.10 there are $\mathbf{p}^* \in \Delta$ and $\mathbf{z}^* \in \zeta_{\mathbf{c}}(\mathbf{p}^*)$ such that $\mathbf{z}^* \leq \mathbf{0}$.
Let $\mathbf{y} = \mathbf{z}^* + \mathbf{s}$. Then $\mathbf{y} \in \mathbf{d}_{\mathbf{c}}(\mathbf{p}^*)$ so there are $\mathbf{x}_1^*, \ldots, \mathbf{x}_m^*$ such that $\mathbf{x}_h^* \in \hat{\xi}_{\mathbf{c},h}(\mathbf{p}^*)$ and $\mathbf{y} = \sum_{h=1}^m \mathbf{x}_h^* \leq \mathbf{s}$. and $\mathbf{y} \leq \mathbf{s}$ because $\mathbf{z}^* \leq 0$.
Also, $\mathbf{x}_h^* \geq \mathbf{0}$ for all h so $\mathbf{x}_h^* \leq \mathbf{s}$. If we choose $\mathbf{c} \gg \mathbf{s}$ then $\mathbf{x}_h^* \ll \mathbf{c}$.
$\mathbf{x}_h^* \in \xi_{\mathbf{c},h}(\mathbf{p}^*)$, i.e. $\mathbf{x}_h^*$ maximises u_h over $B_{\mathbf{c}}(\mathbf{p}^*, \mathbf{p}^* \cdot \overline{\mathbf{x}}_h)$.
Let $N = \left\{\mathbf{x} \in \mathbf{R}_+^n \colon \mathbf{x} \ll \mathbf{c}\right\}$. Then N is an open neighbourhood of $\mathbf{x}_h^*$ in $\mathbf{R}_+^n$.
$\mathbf{x}_h^*$ maximises u_h over $N \cap B(\mathbf{p}^*, \mathbf{p}^* \cdot \overline{\mathbf{x}}_h)$, so by Proposition 7.4 it maximises u_h over all of $B(\mathbf{p}^*, \mathbf{p}^* \cdot \overline{\mathbf{x}}_h)$.
Also, $\mathbf{p}^* \cdot \mathbf{x}_h^* = \mathbf{p}^* \cdot \overline{\mathbf{x}}_h$, hence $\mathbf{p}^* \cdot \mathbf{y} = \mathbf{p}^* \cdot \mathbf{s}$.

Proof of Theorem 7.11 concluded

If $p_i^* = 0$ then we could increase $u_h(\mathbf{x}^*)$ while continuing to satisfy the budget constraint, by increasing x_{hi}^* while leaving x_{hj}^* constant for all $j \neq i$.
Since this can't happen, we know that $p_i^* > 0$ for all i.
We've already seen that $\mathbf{y} \leq \mathbf{s}$ and $\mathbf{p}^* \cdot \mathbf{y} = \mathbf{p}^* \cdot \mathbf{s}$, so $\mathbf{y} = \mathbf{s}$.
In other words,

$$\sum_{h=1}^{m} \mathbf{x}_h^* = \sum_{h=1}^{m} \overline{\mathbf{x}}_h.$$

This completes the proof of Theorem 7.11.
Note that the idea of rationing was useful for technical reasons, but ultimately doesn't matter. None of the rationing constraints are active in the solution.
For a bit more about excess demand, see Sections 7.9 and 7.10.