MAU34802

Lecture 22

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Rescuing hemicontinuity

One way to get a lower and upper hemicontinuous budget correspondence is to restrict its domain. The following proposition from the notes follows, in the case $n = 2$, from the analysis we did last time.

Proposition 7.2 *Let n be a positive integer, and let $B\colon \mathbf{R}^n_+ \times \mathbf{R}_+ \rightrightarrows \mathbf{R}^n_+$ be the budget correspondence that assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbf{R}^n_+ \times \mathbf{R}_+$ the subset $B(\mathbf{p}, w)$ of $\mathbf{R}^n_+$ defined such that*

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbf{R}^n \colon \mathbf{x} \geq 0 \text{ and } \mathbf{p} \cdot \mathbf{x} \leq w\}.$$

Then the budget correspondence $B\colon \mathbf{R}^n_+ \times \mathbf{R}_+ \rightrightarrows \mathbf{R}^n_+$ is both upper hemicontinuous and lower hemicontinuous on the set Γ^n, where

$$\Gamma^n = \{(\mathbf{p}, w) \in \mathbf{R}^n \times \mathbf{R} \colon \mathbf{p} \gg \mathbf{0} \text{ and } w > 0\}.$$

Moreover the subset $B(\mathbf{p}, w)$ of $\mathbf{R}^n_+$ is non-empty, compact, and convex for all $(\mathbf{p}, w) \in \Gamma^n$.

The case of general n is proved in the same way as $n = 2$.

The restriction to $w > 0$ appears to be unnecessary.

Rescuing hemicontinuity, continued

If we want to rescue upper hemicontinuity without imposing positivity conditions on $\mathbf{p}$ and w then we can do it, if we're willing to place upper bounds on $\mathbf{x}$ instead.

The arguments showing that B is not upper hemicontinuous at points where one of p_1 and p_2 is zero and the other is positive used the behaviour of $f(x)$ for large x, so it's plausible that preventing x from becoming large will fix the problem.

That this is in fact the case is the main content of the following theorem.

Proposition 7.1 *Suppose* $\mathbf{c} \in \mathbf{R}^n_+$ *and* $\mathbf{c} \gg \mathbf{0}$. *Define* $B_{\mathbf{c}} \colon \mathbf{R}^n_+ \times \mathbf{R}_+ \rightrightarrows \mathbf{R}^n_+$ *by*

$$B_{\mathbf{c}}(\mathbf{p}, w) = \left\{ \mathbf{x} \in \mathbf{R}^n_+ \colon \mathbf{x} \leq \mathbf{c}, \mathbf{p} \cdot \mathbf{x} \leq w \right\}.$$

Then $B_{\mathbf{c}}$ *is non-empty valued, compact valued, convex valued and upper hemicontinuous, and its restriction to the set*

$$\hat{\Gamma}^n = \left\{ (\mathbf{p}, w) \in \mathbf{R}^n_+ \times \mathbf{R} \colon w > 0 \right\}$$

is lower hemicontinuous.

Rationing

Economically the assumption $\mathbf{x} \leq \mathbf{c}$, i.e. that $x_i \leq c_i$ for all i, describes a situation in which all goods are rationed. So a household can buy at most c_i of the i'th good, no matter how large their wealth w is.
In fact, if $\mathbf{p} \cdot \mathbf{c} < w$ then they won't even be able to spend all their money.
We might be interested in rationing as a real economic phenomenon but in most cases it's a technical assumption to avoid a loss of upper hemicontinuity.
It will ultimately be a harmless assumption because we're modelling a pure exchange economy so there's a fixed amount of each good available and we can always choose c_i greater than the total amount of the i'th good.

Maximising linear functions on the standard simplex

The following fact will be used in the proof of Theorem 7.10.

Proposition 7.3 *Define* $\Delta \subseteq \mathbf{R}^n_+$, $\gamma \colon \mathbf{R}^n \to \mathbf{R}$ *and* $\mu \colon \mathbf{R}^n \rightrightarrows \Delta$ *by*

$$\Delta = \left\{ \mathbf{p} \in \mathbf{R}^n \colon \mathbf{p} \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}$$

$$\gamma(\mathbf{x}) = \max_{1 \leq i \leq n} x_i, \qquad \mu(\mathbf{x}) = \{ \mathbf{p} \in \Delta \colon \mathbf{p} \cdot \mathbf{x} = \gamma(\mathbf{x}) \}.$$

Then μ *is non-empty valued, compact valued, convex valued upper hemicontinuous and* $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p}' \cdot \mathbf{x} = \gamma(\mathbf{x})$ *for all* $\mathbf{p} \in \Delta$, $\mathbf{p}' \in \mu(\mathbf{x})$.

The notes give a direct proof but you can also get everything except the fact that μ is convex valued from Berge's maximum theorem.

For that we can use the fact that the set of maximisers of a linear function on a convex set is always convex.

Utility functions

We already met utility functions in the theory of Nash equilibria. There they were functions on a player's strategy set which satisfied two conditions:
- continuity
- quasiconcavity: $u((1-t)\mathbf{x}+t\mathbf{y}) \geq \min(u(\mathbf{x}), u(\mathbf{y}))$ for $0 < t < 1$.

In the theory of consumer preferences the domain is the set $\mathbf{R}_+^n$ of goods and there is one additional condition:
- strictly increasing: If $\mathbf{x} < \mathbf{y}$ then $u(\mathbf{x}) < u(\mathbf{y})$.

Note: The standard notation in mathematical economics is that $\mathbf{x} < \mathbf{y}$ means $x_i \leq y_i$ for all i and $x_i < y_i$ for some i.
It doesn't mean $x_i < y_i$ for all i. That's denoted by $\mathbf{x} \ll \mathbf{y}$.
Sometimes it's useful to assume that utility functions are strictly quasiconcave, i.e. that $u((1-t)\mathbf{x}+t\mathbf{y}) > \min(u(\mathbf{x}), u(\mathbf{y}))$ for $0 < t < 1$.
Except where explicitly stated we just assume quasiconcavity though.

More about quasiconvexity/quasiconcavity

We used the following theorem on quasiconvex and quasiconcave functions in the previous chapter:

Lemma 6.2 *$f^*((-\infty, b])$ is convex if f is quasiconvex and $f^*([a, \infty))$ is convex if f is quasiconcave.*

I didn't mention there, and we won't need here, the fact that the converse is also true, but it's worth noting since it's usually the easiest way to prove quasiconvexity or quasiconcavity.

Lemma *If f is such that $f^*((-\infty, b])$ is convex for all b then f is quasiconvex. If f is such that $f^*([a, \infty))$ is convex for all a then f is quasiconcave.*

The proof is straightforward. I'll just consider the second statement. For any $\mathbf{x}$ and $\mathbf{y}$ take $a = \min\{f(\mathbf{x}), f(\mathbf{y})\}$. Then $f(\mathbf{x}) \in [a, \infty)$ and $f(\mathbf{y}) \in [a, \infty)$ so $\mathbf{x}, y \in f^*([a, \infty))$ and therefore, by convexity, $(1 - t)\mathbf{x} + t\mathbf{y} \in f^*([a, \infty))$ for all $t \in (0, 1)$, which just means $f((1 - t)\mathbf{x} + t\mathbf{y}) \geq a = \min\{f(\mathbf{x}), f(\mathbf{y})\}$.

Local vs global maxima

A (global) maximiser for a function u is an $\mathbf{x}$ such that $u(\mathbf{x}) \geq u(\mathbf{y})$ for all $\mathbf{y}$.
A local maximiser for a function u is an $\mathbf{x}$ such that there is a neighbourhood V of $\mathbf{x}$ such that $u(\mathbf{x}) \geq u(\mathbf{y})$ for all $\mathbf{y} \in V$.
A global maximiser is always a local maximiser, but a local maximiser needn't be a global maximiser.
For functions on a convex set any local maximum of a strictly quasiconcave function is a global maximum. For functions which are merely quasiconcave this can fail.
$f(x) = \max\{0, x\}$ is quasiconcave on the interval $[-1, 1]$. Its only global maximiser is 1 but every point in $[-1, 0)$ is a local maximiser.

Local and global utility maximisers

Consider a utility function u on the set $B(\mathbf{p}, w)$ where $\mathbf{p} > \mathbf{0}$ and $w > 0$.
Remember, this B is the budget correspondence not the ball.
Even though we assumed quasiconcavity rather than strict quasiconcavity for u local maximisers of u are still global maximisers.
The reason for this is that assumption that u is strictly increasing means that local maximisers always have $\mathbf{p} \cdot \mathbf{x} = w$, even though the definition of $B(\mathbf{p}, w)$ allows $\mathbf{p} \cdot \mathbf{x} < w$
Both of these facts are contained in Proposition 7.4 in the notes.

Individual demand

The indirect utility function of a household is the maximum utility they can attain for given prices and wealth, as a function of those prices and wealth. Their demand correspondence gives the affordable bundles of goods which achieve that maximum. The following theorem describes that function and correspondence, assuming that prices and wealth are positive.

Proposition 7.5 *Suppose $u\colon \mathbf{R}^n_+ \to \mathbf{R}$ is continuous, strictly increasing and quasiconcave. Define $V\colon \Gamma^n \to \mathbf{R}$ and $\xi\colon \Gamma^n \rightrightarrows \mathbf{R}^n_+$ by*

$$V(\mathbf{p}, w) = \max_{\mathbf{x} \in B(\mathbf{p}, w)} u(\mathbf{x}),$$

$$\xi(\mathbf{p}, w) = \{\mathbf{x} \in B(\mathbf{p}, w)\colon u(\mathbf{x}) = V(\mathbf{p}, w)\}.$$

Then V is continuous and ξ is non-empty valued, compact valued, convex valued and upper hemicontinuous.

As with Proposition 7.3, everything follows from the Berge maximum theorem except the fact that $\xi(\mathbf{p}, w)$ is convex, which follows from the quasiconcavity of u.

The demand correspondence with rationing

Just as Proposition 7.2 had an alternate version, 7.1, where we weakened $\mathbf{p} \gg \mathbf{0}$ to $p \geq \mathbf{0}$ but added the condition $\mathbf{x} \leq \mathbf{c}$, so Proposition 7.5 also has an alternate version.

Proposition 7.6 *Suppose* $u\colon \mathbf{R}^n_+ \to \mathbf{R}$ *is continuous, strictly increasing and quasiconcave* $\mathbf{c} \gg \mathbf{0}$. *Define* $\hat{\Gamma}^n \subseteq \mathbf{R}^n_+ \times \mathbf{R}_+$, $V_{\mathbf{c}}\colon \hat{\Gamma}^n \to \mathbf{R}$ *and* $\xi_{\mathbf{c}}\colon \hat{\Gamma}^n \rightrightarrows \mathbf{R}^n_+$ *by*

$$\hat{\Gamma}^n = \left\{(\mathbf{p}, w) \in \mathbf{R}^n_+ \times \mathbf{R}_+ : w > 0\right\},$$

$$V_{\mathbf{c}}(\mathbf{p}, w) = \max_{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)} u(\mathbf{x}),$$

$$\xi_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)\colon u(\mathbf{x}) = V(\mathbf{p}, w)\}.$$

Then $V_{\mathbf{c}}$ *is continuous and* $\xi_{\mathbf{c}}$ *is non-empty valued, compact valued, convex valued and upper hemicontinuous.*

The proof is the same as for Proposition 7.5, except with $\hat{\Gamma}^n$ in place of Γ^n, $B_{\mathbf{c}}$ in place of B, $V_{\mathbf{c}}$ in place of V, $\xi_{\mathbf{c}}$ in place of ξ, and Proposition 7.1 in place of Proposition 7.2.

Addition of compact valued correspondences

If $\Omega \subseteq \mathbf{R}^k$ and $\xi_1, \dots \xi_m \colon \Omega \rightrightarrows \mathbf{R}^n$ then we define $\sum_{h=1}^m \xi_h \colon \Omega \rightrightarrows \mathbf{R}^n$ by saying that $(\sum_{h=1}^m \xi_h)(\mathbf{p})$ is

$$\left\{\mathbf{y} \in \mathbf{R}^n \colon \exists(\mathbf{x}_1, \dots \mathbf{x}_m) \in \xi_1(\mathbf{p}) \times \cdots \times \xi_m(\mathbf{p}) \colon \sum_{h=1}^m \mathbf{x}_h = \mathbf{y}\right\}.$$

Proposition 7.7 *If $\xi_1, \dots, \xi_h$ are non-empty valued, compact valued and upper hemicontinuous then so is $\sum_{h=1}^m \xi_h$.*

The proof is a fairly straightforward application of

Proposition 2.16 *Suppose $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$ and $\Phi \colon X \rightrightarrows Y$ is compact valued. Then Φ is upper hemicontinuous if and only if for every $\mathbf{p} \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\Phi(B_X(\mathbf{x}, \delta)) \subseteq B_Y(\Phi(\mathbf{p}), \epsilon).$$

Note that B's here are balls rather than budget correspondences.