MAU34804

Lecture 21

2026-03-23

David Wilkins

Unfortunately David Wilkins, who developed this module and whose notes we've been using all semester, has died. For anyone interested his funeral will be at 14.15 tomorrow in the Garden Chapel at Mount Jerome Cemetery.

Mock exam

I'll post a mock exam to the module website sometime this week.
I'll post solutions to it next week.
The mock exam is meant to give you a feeling for the types of questions that might appear on the real exam, not the topics covered. The topics for both exams were chosen randomly from the material we've covered, so any overlap in content is coincidental.

Ulimatum game (analysis)

As with other two person games we've considered $S_1 = \Delta_P$, $S_2 = \Delta_Q$, $P\colon \Delta_Q \rightrightarrows \Delta_P$ gives the set of utility maximising responses for the first player to a given strategy from the second player, and $Q\colon \Delta_P \rightrightarrows \Delta_Q$ gives the set of utility maximising responses for the second player to a given strategy from the first player.
The Nash equilibria are the fixed points of the correspondence $M\colon S \rightrightarrows S$ defined by

$$M(\mathbf{p}^*, \mathbf{q}^*) = (P(\mathbf{q}^*), Q(\mathbf{p}^*)).$$

We start by finding the correspondences P and Q. The first player's utility function is

$$u_1(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{m-1} p_j \left(\frac{m-j}{m} \sum_{K\colon j \in K} q_K \right).$$

The $\mathbf{p} \in P(\mathbf{q})$ are those where $p_j = 0$ unless j is a maximiser of $\frac{m-j}{m} \sum_{K\colon j \in K} q_K$.

Analysis, continued

The second player's utility function is

$$u_2(\mathbf{p}, \mathbf{q}) = \sum_K \left(q_K \sum_{j \in K} \frac{j}{m} p_j \right).$$

The quantity $\sum_{j \in K} \frac{j}{m} p_j$ has a maximum possible value of $\sum_{j=1}^{m-1} \frac{j}{m} p_j$, attained when $K = \{1, \ldots, m-1\}$. There may be other K's which attain this maximum though. The condition for K to be a maximiser is that if $p_j > 0$ then $j \in K$.

So the $\mathbf{q} \in Q(\mathbf{p})$ are those for which $q_K = 0$ unless K is a superset of $\{j \colon p_j > 0\}$.

So $(\mathbf{p}, \mathbf{q})$ is a fixed point of M if and only if the following two conditions are satisfied:

- $p_j = 0$ unless j is a maximiser of $\frac{m-j}{m} \sum_{K \colon j \in K} q_K$.
- $q_K = 0$ unless K is a superset of $\{j \colon p_j > 0\}$.

Suppose $(\mathbf{p}, \mathbf{q})$ is a fixed point with $p_i > 0$ and that $i < j$.

Then $q_K > 0$ only if $i \in K$ so $\sum_{K \colon i \in K} q_K = 1$ and

$$\frac{m-i}{m} \sum_{K \colon i \in K} q_K = \frac{m-i}{m}.$$

Analysis, conclusion

The conditions for a Nash equilibrium were that $p_j = 0$ unless j is a maximiser of $\frac{m-j}{m}\sum_{K:\, j\in K} q_K$, and that $q_K = 0$ unless K is a superset of $\{j \colon p_j > 0\}$.
If $p_i > 0$ and $i < j$ then

$$\frac{m-j}{m}\sum_{K:\, j\in K} q_K \le \frac{m-j}{m} < \frac{m-i}{m} = \frac{m-i}{m}\sum_{K:\, i\in K} q_K.$$

So j can't be a maximiser and so by the first condition $p_j = 0$. It follows that there is only one positive p, i.e. that the first player's optimal strategies are all pure.
If $(\mathbf{p}, \mathbf{q})$ is fixed point with $\mathbf{p}$ being the i'th pure strategy then the second condition says that $i \in K$ for all K with $q_K > 0$, while from the first condition we get

$$\frac{m-j}{m}\sum_{K:\, h\in K} q_K \le \frac{m-h}{m}, \qquad \sum_{K:\, h\in K} q_K \le \frac{m-i}{m-h}$$

for all h. This only imposes a restriction when $h < i$.

Comments

The Nash equilibria are the $(\mathbf{p}, \mathbf{q})$ such that $p_i = 1$ for some i and $q_K = 0$ whenever $i \notin K$ and $\sum_{K:\ h \in K} q_K \leq \frac{m-i}{m-h}$ for $h < i$.

Actually, I only proved that all Nash equilibria are of this form but it's easy to check that all points of this form are Nash equilibria.

The set of Nash equilibria is very large, with dimension $2^{m-1} - 1$, and there are lots of weird Nash equilibria.

For example, there's a Nash equilibrium where the first player only offers slices of size $\frac{1}{m}$ and the second player only accepts such slices.

Refusing better offers seems irrational, but it doesn't violate the conditions for a Nash equilibrium since such offers will never be made at this equilibrium.

There's another one where the first player only offers slices of size $\frac{m-1}{m}$ and the second player only accepts such slices.

The first player only ever makes the most generous possible offer, even though there's another equilibrium where the first player always makes the stingiest offer and it's always accepted.

Exchange economies

The rest of the module is concerned with exchange economies.

These are economic models where goods are exchanged, but not produced.

This is complementary to models like Leontief's input-output model, which focus on production, ignoring issues of distribution.

We'll see how individual preferences give rise to supply and demand "curves", and how these intersect at a market-clearing price.

We'll consider a market with m households and n goods.

Neither the word "household" nor the word "good" is to be understood too narrowly.

Quantities of goods are assumed to be non-negative real numbers, ignoring the fact that some are constrained to be integers.

Each household has an initial endowment of goods and its own utility function.

The budget correspondence

$\mathbf{x}$ represents a bundle of goods which can be bought by a household with wealth $w \in \mathbf{R}_+$ at prices $\mathbf{p}$ if and only if $\mathbf{p} \cdot \mathbf{x} \leq w$, i.e if and only if $\mathbf{x} \in B(\mathbf{p}, w)$, where

$$B(\mathbf{p}, w) = \left\{\mathbf{x} \in \mathbf{R}^n_+ \colon \mathbf{p} \cdot \mathbf{x} \leq w\right\}.$$

Do not confuse this with the ball of radius w about $\mathbf{p}$!

What can we say about the hemicontinuity properties of the correspondence $B \colon \mathbf{R}^n_+ \times \mathbf{R}_+ \rightrightarrows R^n_+$?

Consider the case $n = 2$. Then

$$B((p_1, p_2), w) = \begin{cases} \mathbf{R}^2_+ & \text{if } p_1 = 0, p_2 = 0 \\ \{0\} \times \mathbf{R}_+ & \text{if } p_1 > 0, p_2 = 0, w = 0 \\ \mathbf{R}_+ \times \{0\} & \text{if } p_1 = 0, p_2 > 0, w = 0 \\ \{(0,0)\} & \text{if } p_1 > 0, p_2 > 0, w = 0 \\ [0, w/p_1] \times \mathbf{R}_+ & \text{if } p_1 > 0, p_2 = 0, w > 0 \\ \mathbf{R}_+ \times [0, w/p_2] & \text{if } p_1 = 0, p_2 > 0, w > 0 \\ \{(x_1, x_2) \in \mathbf{R}^2_+ \colon \frac{p_1}{w} x_1 + \frac{p_2}{w} x_2 \leq 1\} & \text{if } p_1 > 0, p_2 > 0, w > 0 \end{cases}$$

Is the budget correspondence hemicontinuous? (1/5)

The simplest case is the last one from the previous slide, where $p_1 > 0$, $p_2 > 0$, and $w > 0$, since every nearby point satisfies the same conditions.
The value of B is the triangle with vertices $(0, 0)$, $(0, w/p_2)$, and $(w/p_1, 0)$.
If V is an open set containing this triangle then it also contains the triangles for all nearby values of p_1, p_2, and w, so B is upper hemicontinuous here.
Similarly, if V is an open set with non-empty intersection with the triangle then it contains an interior point of the triangle and this point is also in the corresponding triangles for all nearby values of p_1, p_2, and w, so B is lower hemicontinuous here.
What about points where $p_1 = 0$, $p_2 > 0$, and $w > 0$?
In this case the value of B is $\mathbf{R}_+ \times [0, w/p_2]$.
At some nearby points it is of the same type, but there are points arbitrarily close where $p_1 > 0$ and the value is a triangle.
It's still true that if an open set V intersects the value of B at our point then it intersects its interior and so the intersection with the value at any nearby point is also non-empty, so we still have lower hemicontinuity.

Is the budget correspondence hemicontinuous? (2/5)

Still considering the points where $p_1 = 0$, $p_2 > 0$, and $w > 0$, where the value of B is the half-strip $\mathbf{R}_+ \times [0, w/p_2]$, there are open sets which contain this set, but not all nearby points do.

Consider, for example,

$$V = \{(x_1, x_2) \in \mathbf{R}^2_+ \colon x_2 < w/p_2 + f(x_1)\},$$

where f is a positive continuous function with $\lim_{x\to\infty} f(x) = 0$.

If we leave $p_1 = 0$ and either decrease p_2 slightly or increase w slightly we'll get a slightly larger half-strip which is no longer contained in V, not matter how small the change to p_2 or w is.

So B is not upper hemicontinuous here.

Similar remarks apply the the points where $p_1 > 0$, $p_2 = 0$, and $w > 0$. B is lower hemicontinuous at such points but not upper hemicontinuous.

Is the budget correspondence hemicontinuous? (3/5)

Now consider the case $p_1 > 0$, $p_2 > 0$, and $w = 0$, where the value of B is $\{(0,0)\}$. V has non-empty intersection with this set if and only if $(0,0) \in V$. But $(0,0) \in B(\mathbf{p}, w)$ for all $\mathbf{p}$ and w so $B(\mathbf{p}, w)$ will have non-empty intersection with V for all $\mathbf{p}$ and w, so B is lower-hemicontinuous here.
At nearby points the value of B is either also $\{(0,0\}$, if $w = 0$ at those points, or is a small triangle, if $w > 0$ there. For any open V containing $(0,0)$ this triangle will be contained in V, so we also have upper hemicontinuity.
Now consider the case $p_1 = 0$, $p_2 > 0$, and $w = 0$, where the value of B is $\mathbf{R}_+ \times \{0\}$. B is not lower hemicontinuous at such points. The open set $V = (0, \infty) \times \mathbf{R}_+$ has non-empty intersection with this set but there are points arbitrarily nearby where w is still 0 but $p_1 > 0$. At those points the value of B is $\{0,0\}$, which does not intersect V.
B isn't upper hemicontinuous at these points either. With f as before we take $V = \{(x_1, x_2) \in \mathbf{R}^2_+ : x_2 < f(x_1)\}$ and note that there are nearby points where p_1 is still 0 but w is positive and for these points the value of B is not contained in V.

Is the budget correspondence hemicontinuous? (4/5)

Similarly, B is neither lower nor upper hemicontinuous at the points where $p_1 > 0$, $p_2 = 0$, and $w = 0$.

The remaining case is $p_1 = 0$ and $p_2 = 0$, where the value of B is $\mathbf{R}^2_+$.

B must be upper hemicontinuous here because the only open V which contains $\mathbf{R}^2_+$ is $\mathbf{R}^2_+$ and it contains the value of B at all other points.

Whether B is lower hemicontinuous here depends on whether w is positive. In either case every non-empty open subset intersects $\mathbf{R}^2_+$ so the condition we need to check is whether for every non-empty open set all the nearby points have values of B which intersect that set.

If $w = 0$ this is not the case. We can take $V = (0, \infty) \times (0, \infty)$ and nearby values with $p_1 > 0$ or $p_2 > 0$ as a counterexample.

On the other hand, if $w > 0$ then we can choose some $(x_1, x_2) \in V$ with $x_1 > 0$ and $x_2 > 0$ and the value of B at all sufficiently nearby points will contain (x_1, x_2).

Is the budget correspondence hemicontinuous? (5/5)

In summary, the points where B is upper hemicontinuous are the ones where the p's are either both zero or both negative and the points where it's lower hemicontinuous are the ones where $w > 0$ or p_1 and p_2 are both positive. The points where it's both upper and lower hemicontinuous are the points where p_1 and p_2 are both positive, or $w > 0$ and they're both zero.