MAU34804

Lecture 19

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Simplified Poker

No form of poker that real people play is computationally tractable without heuristics, but Harold Kuhn analysed a simplified version in 1950.

There are two players and three cards, with each player getting one card. The ante is 1 and all bets are 1. Players alternately bet or pass, with betting ending after two bets or two passes or a bet followed by a pass, so the possible courses of betting are pass/pass, pass/bet/pass, pass/bet/bet, bet/pass, and bet/bet.

A player who responds to a bet by passing, i.e. the first player for pass/bet/pass or the second for bet/pass, loses. In the other three cases the player with the higher numbered card wins.

We're only interested in the net winnings, i.e. the part of the pot contributed by the losing player, which is 2 if the other player has bet, as in the pass/bet/bet or bet/bet cases, and 1 otherwise.

Pure strategies

For each of the three cards the first player could decide to pass initially and then pass (fold) in response to a possible bet by the second player, pass initially and then and bet (call) in response, or bet initially. We assign the numbers 0, 1, 2 to these options. If $x_i \in \{0, 1, 2\}$ is the option we choose when holding card i then $9x_1 + 3x_2 + x_3 + 1$ is a number between 1 and 27 from which we can determine each x_i.

For example, 20 means $x_1 = 2$, $x_2 = 0$, and $x_3 = 1$, i.e. bet on 1, always pass on 2, and wait for the second round to bet on 3.

For each of the three cards the second player could decide to pass no matter what the first player does, to pass in response to a pass and bet (raise) in response to a bet, to bet in response to a bet but pass (fold) in response to a bet, or to bet no matter what the first player does. and bet (call) in response, or bet initially. We assign the numbers 0, 1, 2, 3 to these options. If $y_j \in \{0, 1, 2, 3\}$ is the option we choose when holding card j then $16y_1 + 4y_2 + y_3 + 1$ is a number between 1 and 64 from which we can determine each y_j.

For example, 34 means $y_1 = 2$, $y_2 = 0$, and $y_3 = 1$, i.e. bet in response to a pass with a 1 or in response to a bet with a 3 and pass in all other cases.

Mixed strategies, payoff matrix

The mixed strategies involve each player assigning probabilities to each of their 27 or 64 pure strategies.

Each entry is the payoff matrix is computed by checking happens for each of the six possible deals and average them.

$M_{20,34}$ is the expected payoff when the two players choose the strategies from the previous slide: the first player bets on 1, always passes on 2, and waits for the second round to bet on 3, while the second player bets in response to a pass with a 1 or in response to a bet with a 3 and passes in all other cases.

The possible deals are $(1,2)$, $(1,3)$, $(2,1)$, $(2,3)$, $(3,1)$, and $(3,2)$.

The sequence of events these cases are bet/pass, with the first player winning 1, bet/bet, with the second player winning 2, pass/bet/pass, with the second player winning 1, pass/pass, with the second player winning 1, pass/bet/bet, with the first player winning 2, and pass/pass, with the first player winning 1.

Averaging, the expected payoff is 0.

The remaining 1727 entries are computed similarly.

Eliminating bad strategies

A 27 × 64 matrix won't fit on a slide but most pure strategies are obviously stupid and should be assigned probability 0. For example, strategies 1, 4, 7, 10, 13, 16, 19, 22, and 25 for the first player all lead to them losing with a 3, the highest card.

Be careful. Not everything obvious is true! It might seem obvious you should always bet with a 3, but it's not true. Still, Kuhn finds that only strategies 2, 3, 5, 6, 20, 21, 23 and 24 are viable for the first player and 4, 8, 36, and 40 for the second player.

This leads to the more tractable reduced payoff matrix

$$\frac{1}{6}\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -2 & 0 & -3 \\ 1 & -1 & -1 & -3 \\ 0 & -3 & 2 & -1 \\ 0 & -2 & 1 & -1 \end{pmatrix}.$$

Optimal strategies

The von Neumann theorem tells us there is a $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta_P \times \Delta_Q$ such that for all $(\mathbf{p}, \mathbf{q}) \in \Delta_P \times \Delta_Q$

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}).$$

It doesn't tell us there's a unique such $(\mathbf{p}^*, \mathbf{q}^*)$, and in this case there isn't.
There are infinitely many $(\mathbf{p}^*, \mathbf{q}^*)$, all with $f(\mathbf{p}^*, \mathbf{q}^*) = -1/18$, but 24 are basic, in the sense that all of the others are convex combinations of these and none of them is a convex combination of the others.
One basic solution is $p_3^* = 2/5$, $p_5^* = 7/15$, $p_{20}^* = 2/15$, $q_4^* = 2/3$, $q_{40}^* = 1/3$, and all other coordinates are 0.
Note that this assigns a positive probability to pure strategy 20 for the first player, a strategy where initially they bet only on 1, the lowest card!

Comparison with Rock, Paper, Scissors

- Unlike Rock, Paper, Scissors, the optimal strategies are far from unique.
- Like Rock, Paper, Scissors none of the optimal strategies are pure. In other words, none of them are vertices of the strategy simplex.
- Unlike Rock, Paper, Scissors, some responses to an optimal strategy are suboptimal. This is related to the fact that none of the optimal strategies lie in the interior of the strategy simplex.
- Unlike Rock, Paper, Scissors, Simplified Poker has random elements and imperfect information. The von Neumann theorem can still be applied though.

Quasiconvexity and quasiconcavity

The function f from von Neumann's theorem was linear in each of its arguments for fixed values of the other argument.
For generalisations it's useful to weaken that property.
If $K \in \mathbf{R}^n$ is convex then $f\colon K \to \mathbf{R}$ is called *quasiconvex* if

$$f\left((1-t)\mathbf{u} + t\mathbf{v}\right) \leq \max\left(f(\mathbf{u}), f(\mathbf{v})\right)$$

for all $t \in [0,1]$ and is called *quasiconcave* if

$$f\left((1-t)\mathbf{u} + t\mathbf{v}\right) \geq \min\left(f(\mathbf{u}), f(\mathbf{v})\right)$$

$$\min\left(f(\mathbf{u}), f(\mathbf{v})\right) \leq (1-t)f(\mathbf{u}) + tf(\mathbf{v}) \leq \max\left(f(\mathbf{u}), f(\mathbf{v})\right)$$

so convex functions are quasiconvex and concave functions are quasiconcave.
Linear functions are quasiconvex and quasiconcave.
Lemma 6.2 *$f^*((-\infty, b])$ is convex if f is quasiconvex and $f^*([a, \infty))$ is convex if f is quasiconcave.*

Generalising the minimax theorem

How can we generalise von Neumann's theorem?

- Allow *n* players in place of two.
- Replace standard simplices with arbitrary non-empty compact convex subsets of a Euclidean space.
- Replace a single payoff function with continuous utility functions for each player, which they seek to maximise.
- Replace the linearity of the payoff for one player, given the strategy chosen by the other player, with quasiconcavity of the the utility of one player, given the strategies chosen by the other players.

Mathematical formulation

An n-person game, as above, is specified by the following information:
- Non-empty compact convex $S_i \subseteq \mathbf{R}^{m_i}$ for $1 \leq i \leq n$.
- Continuous functions $u_i \colon S \to \mathbf{R}$ for $1 \leq i \leq n$, where $S = S_1 \times \cdots \times S_n$, which are quasiconcave in their i'th argument when the other arguments are fixed.

We will show that for any such S_1, ..., S_n and u_1, ..., u_n there are $\mathbf{x}_1^* \in S_1$, ..., $\mathbf{x}_n^* \in S_n$ such that for each i

$$u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right) \leq u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i^*, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right)$$

Such a $(\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*) \in S$ is called a *Nash equilibrium.*

Theorem 6.3 (Existence of Nash Equilibria) *for every collection of non-empty, compact convex* $S_i \subseteq \mathbf{R}^{m_i}$ *and continuous quasiconcave* $u_i \colon S \to \mathbf{R}$ *for* $1 \leq i \leq n$, *where* $S = S_1 \times \cdots \times S_n$, *there is a Nash equilibrium.*