MAU34804

Lecture 18

2026-03-11

Monotonicity

In the course of the proof we saw that the four conditions are equivalent to the existence of a positive solution $\mathbf{y} \gg \mathbf{0}$ to the equation

$$\mathbf{y} = \mathbf{x} + A\mathbf{y}$$

for each $\mathbf{x} \gg \mathbf{0}$.
Economically, this means we can achieve any level of net production at equilibrium with a sufficently high level of gross production.
Using the monotonicity properties of $(I - A)^{-1}$ proved earlier we can say more:
- Increasing any x requires increasing all of the y's.
- The y's are convex functions of the x's.

The first of these is economically obvious. The second is plausible (diminishing returns) but not really obvious.

The open Leontief model

Last time we considered the closed Leontief model, where every input is also an output. The open Leontief model incorporates *factors of production*, i.e. things which are only inputs, and of which there is a finite supply.
For simplicity we'll assume there's a single factor or production, labour.
Let e_j be the amount of labour required per unit of the j'th good. We'll assume that $\mathbf{e} \gg \mathbf{0}$.
Just as for the closed Leontief model, for the open Leontief model we have a set of equivalent conditions:
- There is a $\mathbf{y} \geq \mathbf{0}$ such that $\mathbf{y} - A\mathbf{y} \gg \mathbf{0}$.
- There is a $\mathbf{p} \geq \mathbf{0}$ such that $\mathbf{p} \geq A^T\mathbf{p} + \mathbf{e}$.
- Every eigenvalue of A has absolute value less than 1.

We saw last time that the first and last conditions are equivalent, and also that they're equivalent to the existence of $\mathbf{p}$ such that $\mathbf{p} \gg A^T\mathbf{p}$, which is implied by the middle condition, so the only thing we need to show is the reverse direction, i.e. that if either of the other conditions hold then so does the middle one.

The open Leontieff model, continued

Our three conditions were
- There is a $\mathbf{y} \geq \mathbf{0}$ such that $\mathbf{y} - A\mathbf{y} \gg \mathbf{0}$.
- There is a $\mathbf{p} \geq \mathbf{0}$ such that $\mathbf{p} \geq A^T\mathbf{p} + \mathbf{e}$.
- Every eigenvalue of A has absolute value less than 1.

Under the last condition we showed that $\left(I - A^T\right)^{-1} \gg O$ so

$$\mathbf{p} = \left(I - A^T\right)^{-1}\mathbf{e}$$

works.

At these prices every production activity exactly breaks even.

The prices $\mathbf{p}$ represent the labour cost of producing each good, taking into account indirect costs from the other goods needed to produce it.

If you know any Marx, this looks a lot like the labour theory of value.

Zero sum two person games

We consider games between two players, each of whom has a finite number of pure stategies available. The net payment from one player to the other is a function of the pure strategies each selects.

There are no other players or payments, so the sum of the net payment from Player A to Player B and the net payment from Player B to Player A is zero.

In addition to the pure strategies players can adopt mixed strategies, selecting a pure strategy at random with specified probabilities. A mixed strategy is characterised by those probabilities.

We assume the goal of each player is to maximise the expected value of the net payment they receive.

That's not an obvious assumption. It doesn't account for risk aversion, for example.

Mathematical formulation

Let m and n be the numbers of pure strategies available to Players A and B respectively. Fix an ordering of those strategies.

A mixed strategy for Player A is then an element of the $m-1$-simplex

$$\Delta_P = \left\{ \mathbf{p} \in \mathbf{R}^m \colon p_1 \geq 0, \ldots, p_m \geq 0, \sum_{i=1}^{m} p_i = 1 \right\},$$

where p_i is to be interpreted as the probability that the player selects the i'th pure strategy.

Similarly, Player B's mixed strategies are elements of

$$\Delta_Q = \left\{ \mathbf{q} \in \mathbf{R}^m \colon q_1 \geq 0, \ldots, q_n \geq 0, \sum_{j=1}^{n} q_j = 1 \right\}.$$

Note that pure strategies are also mixed strategies, just with all but one of their probabilities equal to zero. They correspond to the vertices of the simplices.

Optimal response

If a_{ij} is the net payment from Player B to Player A when Player A selects their i'th pure strategy and Player B selects their j'th then the expected net payment when they choose mixed strategies $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$ is

$$f(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \mathbf{p}^T A \mathbf{q}.$$

Suppose Player A has to choose their mixed strategy first, and this choice is revealed to Player B.
If Player A chooses $\mathbf{p} \in \Delta_P$ then the best expected outcome Player B can achieve is

$$\mu_Q(\mathbf{p}) = \min_{\mathbf{q} \in \Delta_Q} f(\mathbf{p}, \mathbf{q}).$$

The minimum exists by the Extreme Value Theorem.
To achieve it they should choose a mixed strategy $\mathbf{q}$ from

$$Q(\mathbf{p}) = \{\mathbf{q} \in \Delta_Q \colon f(\mathbf{p}, \mathbf{q}) = \mu_Q(\mathbf{p})\}.$$

Payoff from optimal strategies

Working backwards, if the Player A knows that Player B will respond optimally then they should choose a mixed strategy $\mathbf{p}$ which maximises $\mu_Q(\mathbf{p})$, to achieve an expected net payment of

$$\max_{\mathbf{p}\in\Delta_P} \mu_Q(\mathbf{p}) = \max_{\mathbf{p}\in\Delta_P} \min_{\mathbf{q}\in\Delta_Q} f(\mathbf{p},\mathbf{q}).$$

Similarly, if Player B chooses first then Player A should respond to $\mathbf{q}$ with $\mathbf{p} \in Q(\mathbf{p})$ where

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P \colon f(\mathbf{p},\mathbf{q}) = \mu_P(\mathbf{q})\}, \quad \mu_P(\mathbf{q}) = \max_{\mathbf{q}\in\Delta_Q} f(\mathbf{p},\mathbf{q}).$$

Knowing this, Player B chooses $\mathbf{q}$, achieving the expected payment

$$\min_{\mathbf{q}\in\Delta_P} \mu_P(\mathbf{q}) = \min_{\mathbf{q}\in\Delta_Q} \max_{\mathbf{p}\in\Delta_P} f(\mathbf{p},\mathbf{q}).$$

Von Neumann's Minimax Theorem

General properties of maxima and minima ensure that

$$\max_{\mathbf{p}\in\Delta_P}\min_{\mathbf{q}\in\Delta_Q} f(\mathbf{p},\mathbf{q}) \le \min_{\mathbf{q}\in\Delta_Q}\max_{\mathbf{p}\in\Delta_P} f(\mathbf{p},\mathbf{q}).$$

In other words, there is nothing to be gained by being forced to choose your strategy first.

Remarkably though, if both players play optimally there's no penalty to being forced to go first either.

Theorem 6.1 (Von Neumann's Minimax Theorem) *With Δ_P, Δ_Q and f defined as above,*

$$\max_{\mathbf{p}\in\Delta_P}\min_{\mathbf{q}\in\Delta_Q} f(\mathbf{p},\mathbf{q}) = \min_{\mathbf{q}\in\Delta_Q}\max_{\mathbf{p}\in\Delta_P} f(\mathbf{p},\mathbf{q}).$$

In fact there is a $(\mathbf{p}^*,\mathbf{q}^*)\in\Delta_P\times\Delta_Q$ such that for all $(\mathbf{p},\mathbf{q})\in\Delta_P\times\Delta_Q$

$$f(\mathbf{p},\mathbf{q}^*) \le f(\mathbf{p}^*,\mathbf{q}^*) \le f(\mathbf{p}^*,\mathbf{q}).$$

Interpretation

What does

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}).$$

mean?

From Player A's point of view,

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*)$$

means that they'll be at least as well off choosing $\mathbf{p}^*$ as any other available strategy, assuming that Player B chooses $\mathbf{q}^*$.

But

$$f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}).$$

means that they'll be no worse off if Player B chooses some other available strategy, assuming they've chosen $\mathbf{p}^*$.

From Player B's point of view things are similar, switching the roles of the players and their strategies and switching the roles of the two inequalities.

Proof of von Neumann's theorem

By Berge's Maximum Theorem (2.23) the correspondences P and Q are non-empty valued, compact valued, convex valued and upper hemicontinuous.
The same is then true of the correspondence $\Phi\colon \Delta_P \times \Delta_Q \rightrightarrows \Delta_P \times \Delta_Q$ defined by

$$\Phi(\mathbf{p}, \mathbf{q}) = (P(\mathbf{q}), Q(\mathbf{p})).$$

Every compact subset of $\mathbf{R}^{m+n}$ is closed, so Φ is closed valued and hence, by Proposition 2.11, $\mathrm{Graph}(\Phi)$ is closed.
Also, products of convex sets are convex.
By the Kakutani Fixed Point Theorem (5.4), there is a $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta_P \times \Delta_Q$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in \Phi(\mathbf{p}^*, \mathbf{q}^*)$, i.e. $\mathbf{p}^* \in P(\mathbf{q}^*)$ and $\mathbf{q}^* \in Q(\mathbf{p}^*)$.
The first of these statements implies $f(\mathbf{p}, \mathbf{q}^*) \le f(\mathbf{p}^*, \mathbf{q}^*)$ for all $\mathbf{p} \in \Delta_P$, while the second implies $f(\mathbf{p}^*, \mathbf{q}^*) \le f(\mathbf{p}^*, \mathbf{q})$. for all $\mathbf{q} \in \Delta_Q$.
This proof is non-constructive but one can give constructive proofs using, for example, the theory of linear programming, the subject of the other MAU3480x module.
The algorithms for finding $(\mathbf{p}^*, \mathbf{q}^*)$ can be slow in theory but are fast in practice.

Rock, Paper, Scissors

We already considered Rock, Paper, Scissors informally but we can now treat it with von Neumann's theorem.

Let p_1, p_2, and p_3 be the probabilities for Rock, Paper, and Scissors for the first player and let q_1, q_2, and q_3 be the probabilities for the second player.

The payoff matrix is

$$M = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

and the payoff function is

$$f(\mathbf{p}, \mathbf{q}) = -p_1 q_2 + p_1 q_3 - p_2 q_3 + p_2 q_1 - p_3 q_1 + p_3 q_2$$

Rock, Paper, Scissors, continued

Von Neumann tells us there are $\mathbf{p}^*$ and $\mathbf{q}^*$ such that $f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$ for all $\mathbf{p} \in \Delta_P$ and all $\mathbf{q} \in \Delta_Q$.
In particular, the combined inequality holds for

$$(p_1, p_2, p_3) = (q_3^*, q_1^*, q_2^*), \quad (q_1, q_2, q_3) = (p_3^*, p_1^*, p_2^*).$$

This gives

$$-q_3^* q_2^* + q_3^* q_3^* - q_1^* q_3^* + q_1^* q_1^* - q_2^* q_1^* + q_2^* q_2^* \leq -p_1^* p_1^* + p_1^* p_2^* - p_2^* p_2^* + p_2^* p_3^* - p_3^* p_3^* + p_3^* p_1^*$$

$$\frac{(q_1^* - q_2^*)^2}{2} + \frac{(q_2^* - q_3^*)^2}{2} + \frac{(q_3^* - q_1^*)^2}{2} \leq -\frac{(p_1^* - p_2^*)^2}{2} - \frac{(p_2^* - p_3^*)^2}{2} - \frac{(p_3^* - p_1^*)^2}{2}$$

so the only optimal set of strategies is the one where $p_1^* = p_2^* = p_3^*$ and $q_1^* = q_2^* = q_3^*$, namely $\mathbf{p}^* = \mathbf{q}^* = (1/3, 1/3, 1/3)$.

Comments on Rock, Paper, Scissors

For Rock, Paper, Scissors there was a unique pair of optimal strategies.
Neither of those strategies was pure.
Also, if your opponent chooses their optimal strategy then your expected payoff is independent of which strategy you choose.
Those three facts are features of this particular example, not general features of two player zero sum games. They holds for some choices of M but not for others.
There are other ways to analyse Rock, Paper, Scissors.
In the first week of term I actually looked at the correspondences P and Q to find the fixed point.
The notes largely follow this lecture, but need to check separately that if $\mathbf{p}^* = \mathbf{q}^* = (1/3, 1/3, 1/3)$ then $f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$ for all $\mathbf{p} \in \Delta_P$ and all $\mathbf{q} \in \Delta_Q$, since von Neumann's theorem hasn't been proved at that point.